3.8 Equations, Equalities and the Laws of Nature

We have seen that in the set of rational numbers all the **basic arithmetic operations**: addition, subtraction, multiplication and division (except for division by zero), are always feasible. The good properties of these operations give us the freedom and simplicity in calculation that we did not have either in the set of natural numbers nor in the set of integers. Now we will use these advantages in solving equations.

Example 3.8.1. After a hard business day, pirate Giorgio decided to relax a bit. He took several gold coins and parked his boat in front of the *Double Luck* casino. The admission cost five gold coins, but in the casino he was lucky and doubled the number of gold coins that he had on him. He paid five gold coins for parking, and left on his boat for another casino *Free Admission*, where he again doubled his amount of gold coins. However, after he had left and paid six gold coins for parking, he had no coins left, and he wondered why he had not one gold coin in his pocket when he had been winning constantly. How many coins did the pirate Giorgio take when he set out to have fun?

Solution. Let us call the required number of unknown gold coins x. In order to determine how much x is, we have to make use of the information mentioned. After paying the admission to the first casino, Giorgio had $x-5$ gold coins. Since in the games of chance he doubled this number, he left the casino with $(x-5)\cdot 2$ gold coins. However, he had to pay 5 gold coins for the parking, so he entered the second casino (where the admission was free) with $(x-5)\cdot 2-5$ gold coins. He had luck there as well, and so he left with double the number of gold coins, which means $[(x-5)\cdot 2-5]\cdot 2$ gold coins. After paying 6 gold coins for the parking, he had nothing left. Thus,

$$[(x-5)\cdot 2-5]\cdot 2-6=0$$

This is the required information about the unknown number x.

Equation with One Unknown

This type of information about an unknown number, a condition in the form of equality, is called an **equation** (with one unknown).

The solution of the equation is any number which fulfils this condition.

To solve an equation means to find all its solutions.

We will deal in more detail with equations and their application in solving actual problems (like the one with the pirate) in Circle 2. For the moment, we will consider the equation as information about an unknown number x, on the basis of which we will try to discover it. However, unlike in the detective novels by Agatha Christie, we will not need any inspired reasoning by detective Hercule Poirot, but rather simple calculation. The aim is, namely, to obtain from the initial information $[(x-5)\cdot 2-5]\cdot 2-6=0$ the information in the form

$$x = \text{something known}$$

This will be realised by a number of simple inferences. Every such inference consists of a simple step – we will add or subtract the same number both from the left and the right side of equation, or we will multiply or divide them by the same non-zero number. Every such step is possible because these operations are feasible in the set of rational numbers. Every one is correct since, by applying the same operation equally on the left and right sides, we will again obtain equal left and right sides. For instance, we can add to the left and right side of the initial equation the number 100. Briefly, we say that we added 100 to the equation and we write down:

$$[(x-5)\cdot 2-5]\cdot 2-6=0 \quad |+100$$

When we add 100 both to the left and the right side, we get

$$[(x-5)\cdot 2-5]\cdot 2-6+100=0+100$$

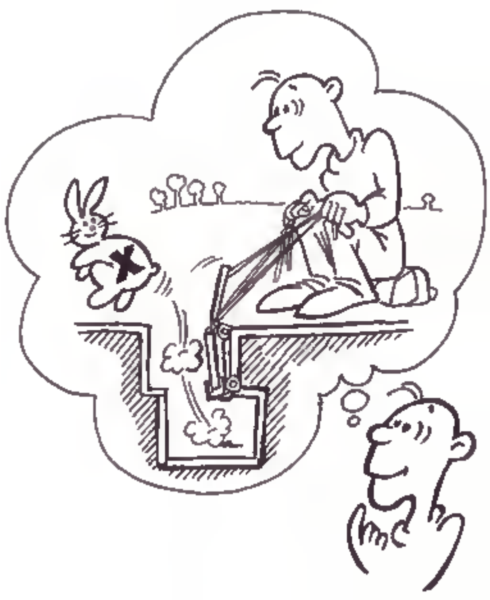
After simplifying both sides, the equation is

$$[(x-5)\cdot 2-5]\cdot 2+94=100$$

Since the step is correct, we have again obtained a true item of information about x. However, the problem is that it is no better than the initial information. The number x is no more known now than before. However, if instead of 100 we add 6 to the equation, we will get better information about the number x, because with this action we will cancel the subtraction of the number 6 on the left side of the equation:

$$[(x-5)\cdot 2-5]\cdot 2-6=0 \quad |+6$$
$$[(x-5)\cdot 2-5]\cdot 2-6+6=0+6$$
$$[(x-5)\cdot 2-5]\cdot 2=6$$

Now we are closer to the number x. Its discovery may be compared to a rabbit in the forest caught in a trap that you want to save by opening the trap. You disassemble the trap from the outside.



We have already cancelled the subtraction of the number 6 by adding the number 6 to the equation. Now we need to cancel the multiplication by the number 2. We will do this by dividing the equation by the number 2:

$$
\begin{aligned}
[(x-5)\cdot 2-5]\cdot 2 &= 6 \quad |:2\\
[(x-5)\cdot 2-5]\cdot 2:2 &= 6:2\\
(x-5)\cdot 2-5 &= 3
\end{aligned}
$$

The strategy for discovering the number x is clear. Although we can perform arbitrary operations with the equation, we will perform the one that will cancel the last operation done with number x. And this is its inverse operation. Thus we will now cancel the subtraction of number 5 by adding the number 5, and so on:

$$
\begin{aligned}
(x-5)\cdot 2-5 &= 3 \quad |+5\\
(x-5)\cdot 2-5+5 &= 3+5\\
(x-5)\cdot 2 &= 8 \quad |:2\\
x-5 &= 4 \quad |+5\\
x &= 9
\end{aligned}
$$

The number x has been discovered! Thus, pirate Giorgio had 9 gold coins in his pocket. Although in every casino he doubled the number of gold coins, he had to pay too much for admissions and parking. Oh my dear Giorgio, there is always a bigger fish. □

We could eliminate all operations precisely owing to the fact that in the set of rational numbers, for every basic arithmetic operation there is its inverse operation, except for multiplication by zero. Not being able to divide by zero is not a restriction in solving equations, since such a situation is very rare (we will deal with this in *Circle 2*). True, we can multiply by zero, but this is completely non-informative. Indeed, if we multiplied the initial equation (or any other) by zero, since multiplication by zero yields zero, we would get $0 = 0$. Therefore, we would not find out anything new, and particularly not about the sought unknown. Simply, by multiplying the equation by zero, we would destroy the information.

Bearing in mind the previous example, let us analyse the process of solving an equation. The permitted steps in solving are:

- *Any description on any side can be replaced by a different description of the same number.* For instance, the description $x+1+2x-3$ can be replaced by the description $3x-2$.
- *With both sides we can do the same*, because if we perform the same operation f to both sides of the equality $L=D$, we will obtain equality again: $f(L)=f(D)$. For instance, if $2x-3=x+1$, if we subtract on both sides x, we will get the equality $2x-3-x=x+1-x$.

These rules are correct and simple, although there are some details that we will leave for *Circle 2*. For the moment, these details will not be significant to us since the only operations that we will do will be addition and subtraction, as well as multiplication and division by a non-zero number. Since we are working in the set of rational numbers, these operations are always feasible. Since every operation has its inverse operation, whenever - by applying one of these operations to one equation - we get another equation, by inverse operation from the second equation we obtain the first one once again. This establishes one significant connection between these equations: *equations are **equivalent**, that is to say they have the same solutions*. Generally, for two conditions we say that they are **equivalent** when the same objects satisfy them. *Thus, solving of the equation is reduced to repeated transformations according to the mentioned rules into equivalent equations, until the final equation is obtained whose solution we can "see".* In the previous example such a final equation was the equation $x=9$. The only number that satisfies this equation is 9. Since the equation $x=9$ is equivalent to the initial equation from the example, $[(x-5)\cdot 2-5]\cdot 2-6=0$, the number 9 is at the same time the only solution of the initial equation. Naturally, in transforming the equations we use the permitted steps with a certain strategy, in order to obtain a simple equation by which we see the solution. The basic principle that we use is the **principle of inverse operation**: *when we want to cancel the last operation on one side of the equation, we apply the inverse operation on the equation.*

Procedure (Restricted) for Solving Equations

Our aim is to get an equation which is equivalent to the initial equation and which gives us a solution. We try to get it by progressively applying the following steps to the initial equation:

- Any description on either side can be replaced by a different description of the same number.
- We may add to or subtract from both the left and the right side of the equation the same number. We may also multiply or divide them with the same non-zero number.

One of the basic strategies of the procedure is the principle of inverse operation: in order to eliminate the last operation performed on one side of the equation, we apply both to the left and the right side of the equation its inverse operation.

Example 3.8.2. Pirate Giorgio, angry at being robbed so dishonourably (without a fight), decided to raid the *Double Luck* casino. In the dead of night he managed to get to the cash register and to break the code (by smashing the cash register with an axe several times). With a full bag of gold coins he started towards the exit, but he bumped into the cashier. The cashier requested two thirds of the treasure to keep quiet. Giorgio gave him two thirds of the treasure and 4 gold coins extra as a tip (a bad habit he acquired by hanging out with waiters). At the very exit of the casino he bumped into the security guard and to prevent him from betraying him, he gave the guard half of the rest of the loot and 2 more gold coins. Just as he was about to board his ship, he bumped into the local gendarme and had to give him three fourths of the rest of the treasure (something like a theft tax) and 8 gold coins more as a tip. When he arrived home, he was disappointed to find that he only had one gold coin left. How many stolen gold coins was Giorgio sorry about?

Solution. Having given $\frac{2}{3}$ of the robbed treasure x to the cashier, he had $\frac{1}{3}$ of the treasure left, and this amount was also reduced by the 4 gold coin tip. Thus, he had $\frac{1}{3}x-4$ gold coins left. In the same way we can determine that after the encounter with the security guard he had $\frac{1}{2}\left(\frac{1}{3}x-4\right)-2$ gold coins, and after meeting the gendarme $\frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{3}x-4\right)-2\right]-8$. Since only one gold coin remained, the required equation for x is:

$$\frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{3}x-4\right)-2\right]-8=1$$

This equation will be solved in the manner already described:

$$
\begin{aligned}
\frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{3}x-4\right)-2\right]-8 &= 1 \quad |+8\\
\frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{3}x-4\right)-2\right] &= 9 \quad |\cdot 4\\
\frac{1}{2}\left(\frac{1}{3}x-4\right)-2 &= 36 \quad |+2\\
\frac{1}{2}\left(\frac{1}{3}x-4\right) &= 38 \quad |\cdot 2\\
\frac{1}{3}x-4 &= 76 \quad |+4\\
\frac{1}{3}x &= 80 \quad |\cdot 3\\
x &= 240
\end{aligned}
$$

If the unknown in the equation can be found in several places, we will keep simplifying the left and the right side of the equation and move the members from one side to the other in order to have the unknown only in one place. The following examples show a typical procedure of solving such equations.

Example 3.8.3. Let us solve:

1. $3-(2x-3)=4(x-1)-[1-2(x-1)]$
2. $\dfrac{1+2(4-x)}{2}-\dfrac{2-x}{3}=\dfrac{6-2x}{4}+\dfrac{5}{2}$

Solution.

1. First, we will simplify each side:

$$\begin{aligned} 3-(2x-3)&=4(x-1)-[1-2(x-1)]\\ 3-2x+3&=4x-4-[1-2x+2]\\ 6-2x&=4x-4-1+2x-2\\ 6-2x&=6x-7 \quad |-6x-6 \end{aligned}$$

Parts with the unknown are moved to one side and the rest to the other:

$$\begin{aligned} -2x-6x&=-7-6\\ -8x&=-13 \quad |:(-8) \end{aligned}$$

By dividing by -8 we will bring x into the clear:

$$x=\frac{13}{8}$$

2. We get rid of the denominator by multiplying the equation by the least common multiple of all the denominators:

$$\begin{aligned} \frac{1+2(4-x)}{2}-\frac{2-x}{3}&=\frac{6-2x}{4}+\frac{5}{2} \quad |\cdot 12\\ \frac{1+2(4-x)}{\cancel{2}}\cdot\cancel{12}^{6}-\frac{2-x}{\cancel{3}}\cdot\cancel{12}^{4}&=\frac{6-2x}{\cancel{4}}\cdot\cancel{12}^{3}+\frac{5}{\cancel{2}}\cdot\cancel{12}^{6}\\ 6\cdot(1+2(4-x))-4\cdot(2-x)&=3\cdot(6-2x)+5\cdot 6 \end{aligned}$$

We simplify both sides:

$$\begin{aligned}6\cdot(1+8-2x)-8+4x &= 18-6x+30\\ 6+48-12x-8+4x &= 18-6x+30\\ -8x+46 &= 48-6x \quad |+6x-46\end{aligned}$$

We transfer the parts:

$$\begin{aligned}-8x+6x &= 48-46\\ -2x &= 2 \quad |:(-2)\\ x &= -1\end{aligned}$$

The obtained solution can be always checked by substituting it in the equation. By including $x=-1$ into the equation we get

$$\frac{1+2(4-(-1))}{2}-\frac{2-(-1)}{3}=\frac{6-2\cdot(-1)}{4}+\frac{5}{2}$$

The calculation yields $\frac{9}{2}=\frac{9}{2}$. We have obtained a true statement. Thus, the number -1 satisfies the condition and it really is the solution of the equation. By verifying the obtained solution we can discover the error in solving the equation. There are also other reasons that require verification from time to time, but we will talk about it later, when we study equations in more detail in *Circle 2*. □

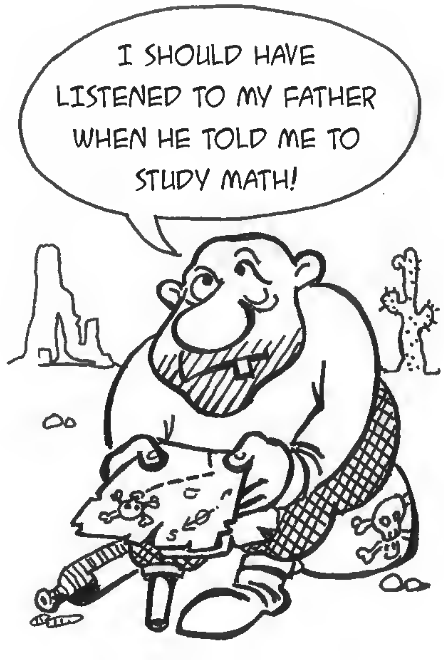
In the software *SageMath* we have the command *solve* for solving equations. The command *solve* has two inputs – we have to specify which equation we want to solve and according to which variable. In this way, we can solve the last equation from the previous example:

```
solve((1+2*(4-x))/2-(2-x)/3==(6-2*x)/4+5/2,x)
# The sign for equality in an equation is written as ==!
```

We will get

```
[x == -1]
```

Example 3.8.4. Since he was left without any cash, pirate Giorgio remembered the treasure map that his late father Domnius had left him. Following the instructions on the back of the map, he sailed to the island of *Fumija* and found a stone with an engraved dead head. When he lifted the stone, he was struck by a current. He did not notice that underneath the dead head there was the inscription *HIGH VOLTAGE! DANGER!*. Thanks to the electric shock he realised that he had made a mistake and soon he found the right stone with its dead head. When he lifted it, underneath he found on impregnated leather the rest of the instructions. They said "*From this place start walking towards the East and make double as many steps as you will make later towards the South, and a hundred steps more. When you make a total of five times more steps than what you made when walking towards the South, the treasure will be under your feet. Love from your dad Domnius.* " After a good cry in remembrance of his dear dad, who was much respected by everyone as a diligent and pious pirate, he started to follow the instructions devotedly. But alas, he could by no means determine how far in which direction he had to go. Finally, he sat down crestfallen on the stone and remembered again his late dad. And it was only then that he understood why his father kept constantly warning him he should learn math, and who was now to blame that he did not listen to his dad? If you happen to arrive on the island of *Fumija* (I will not tell you where this island is so that you will not arrive there before me) and find the engraved dead head (that does not warn of high voltage), how many steps will you take eastwards, and how many southwards in order to find the treasure?



Solution. If we translate this problem into math language, we will get a simple equation. Let x be the number of steps that need to be taken southwards. It follows from the instructions that one should go $2x + 100$ steps eastwards. The instructions also say that the total number of steps eastwards and southwards has to be 5 times greater than x. Hence,

$$x + 2x + 100 = 5x$$

We will solve the equation easily in the manner already described:

$$x + 2x - 5x = -100 \;\rightarrow\; -2x = -100/:(-2) \;\rightarrow\; x = 50$$

Thus, one should take $2x + 100 = 200$ steps eastwards, and then take $x = 50$ more steps southwards. □

Here is (preventively) a little of math against electric shocks.

Example 3.8.5. Which resistor R_2 has to be connected parallel with resistor $R_1 = 20$ Ohm, if we want the total resistance of the parallel connection to be $R = 10$ Ohm?

Solution. We already know the formula for parallel connection of resistance:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

It connects the total resistance R with resistances R_1 and R_2 of parallel connected resistors. In the example 3.6.1 on page 137 we applied it for the calculation of total resistance. We introduced into the formula the known resistances R_1 and R_2 and calculated. Nobody now needs to give us a new formula into which we will introduce R and R_1 in order to calculate R_2. Into the existing formula we simply introduce the known values for R and R_1 and we solve the equation:

$$\frac{1}{10} = \frac{1}{20} + \frac{1}{R_2} \;\rightarrow\; \frac{1}{10} - \frac{1}{20} = \frac{1}{R_2} \;\rightarrow\; \frac{1}{20} = \frac{1}{R_2} \;\rightarrow\; R_2 = 20$$

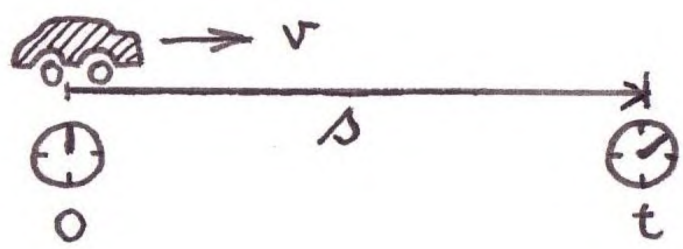
Thus, in order to halve by means of parallel connection the initial resistance of 20 Ohm, the other resistor also has to be 20 Ohm.

Not only is it that nobody has to give us a new formula for calculation, but we can deduce it by ourselves! In the same order in which we have solved the equation, we can express R_2 by means of R and R_1:

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2} \rightarrow \frac{1}{R}-\frac{1}{R_1}=\frac{1}{R_2} \rightarrow \frac{R_1-R}{RR_1}=\frac{1}{R_2} \rightarrow R_2=\frac{RR_1}{R_1-R}$$

Once again we can see that *our operations with variables rather than the names of concrete numbers yield much more*. Not only did we solve the concrete problem but we also found the formula with which we can solve all problems of such a type (the total resistance and one resistor are known, and the other resistor needs to be found). □

Using equalities (formulas) we frequently express the laws according to which the quantities in certain situations are related. As an example, a simple law for a simple situation will be introduced now and we will see how it helps us in connecting the quantities in more complex situations as well. When a car moves uniformly at a certain speed v, it will in time t travel the distance s:



The mentioned values are connected by the following formulas:

$$s=v\cdot t \qquad v=\frac{s}{t} \qquad t=\frac{s}{v}$$

Every formula expresses a value by means of the others. *But all these are variants of one formula, and therefore not everything needs to be remembered*. It is sufficient to remember just one, for example the first:

$$s=v\cdot t$$

If the speed of movement v and the elapsed time t are given, by inserting the values for v and t we will easily calculate the travelled distance. If the elapsed time t and the travelled distance s are known, in order to calculate the speed v, we will insert the known values for s and t into the formula and solve the equation. Or even better, by transforming the formula we will express the speed v by means of the distance s and time t, and then insert the values:

$$s = vt \quad |:t \quad \to \quad \frac{s}{t} = v \quad \to \quad v = \frac{s}{t}$$

Example 3.8.6. If a car travelled at a speed of 80 kilometres per hour, what distance did it travel in $\frac{3}{4}$ hours? How much time would a car need to cover this distance by moving at a speed of 120 kilometres per hour?

Solution. Into the formula for the travelled path $s = v \cdot t$ we will insert speed v = 80 and time $t = \frac{3}{4}$:

$$s = vt = 80 \cdot \frac{3}{4} = 60$$

Thus, the car travelled 60 kilometres.

In order to solve the second part of the problem, we will express t by means of s and v, and then we will insert $s = 60$ and $v = 120$:

$$s = vt \quad |:v \quad \to \quad t = \frac{s}{v}$$

Thus, $t = \frac{60}{120} = \frac{1}{2}$ hours.

The driver who was speeding and thus put himself and others in danger, arrived only 15 minutes before the driver who drove at a moderate speed. It is likely that he spent the time he had saved bragging about having arriving quickly. As my friend Joe would say: in traffic the winner is not the one who arrives earlier but the one who arrives at all.

The second part of the task could be solved by introducing into the initial equation the known values for s and v, and then by solving the equation. However, it is always better to express the required quantity by means of the known ones (by solving the equation in the general manner), and only then to substitute the values. This is how we develop the very powerful ability of thinking with variables and discovering new relations between quantities. □

Procedure of Finding an Unknown Value

When an equality connects some quantities one of which is unknown, the unknown quantity can be found in the following way. We insert into the equality values for the known quantities. We will get an equation and by solving it we will find the unknown quantity. However, it is better to express the unknown quantity by means of the known quantities, using the same procedure which is used to solve the equation, and only then to insert values for the known quantities. In this way we get not only the solution of the concrete problem but also a new equality that solves all the problems of this type.

We will now apply the formula for uniform motion in the analysis of more complex situations.

Example 3.8.7.

1. At the foot of the mountain Romeo was calling after Juliet, but the only answer he got was his echo, and this with $3\frac{1}{2}$ seconds delay. If sound travels at a speed of 330 metres per second, how far were the mountain rocks from Romeo?

2. A train 70 metres long was passing over a bridge at a speed of 20 metres per second. Romeo was bored under the bridge waiting for Juliet. He discovered that the bridge kept vibrating for a full 6 seconds. He was bothered by the questions: Where is Juliet? How long is the bridge? When will Juliet come?

3. When he saw Juliet, Romeo started to run fast and in 2 seconds he had already reached a speed of 6 m per second. What distance did he cover?

Solution.

1. Let us call the unknown distance to the rocks s. Since the speed of sound is $v = 330$ metres per second, and the sound travel time is $t = 3\frac{1}{2} = \frac{7}{2}$ seconds, somebody could be misled by these symbols to make the wrong conclusion that according to the formula for uniform motion $s = v \cdot t$. However, this is not so. *Formulas cannot be formally*

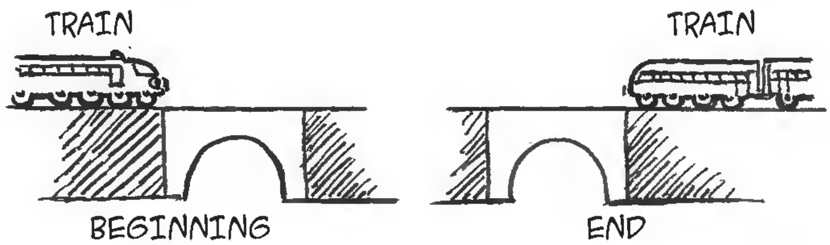
applied. In order to be able to apply a formula correctly, we have to understand its content. We have to understand the meaning of the symbols it contains, that is, the values that it connects. The formula for uniform motion connects the movement speed v, the passed time t and the distance covered during this period of time. In this case this distance is not s but $2s$, since during this period of time the sound travelled to the rocks and returned back to Romeo. Thus, it travelled a distance of 2s. *Only by understanding the formula for uniform motion can we relate correctly the unknown value s with the known values v and t*:

$$2s = v \cdot t$$

It is simple from this equation, dividing by 2, to calculate the required distance from the rocks:

$$s = \frac{1}{2}vt = \frac{1}{2} \cdot 330 \cdot \frac{7}{2} = 577\frac{1}{2} \text{ metres}$$

2. Where Juliet was and when she would arrive were things known only to her. We can help Romeo only regarding the length of the bridge. Formulas are very efficient, but they are not omnipotent. In order to solve the question of the bridge length we will set adequate equations. *Everything spoken about has to be named. Even the known values should be assigned symbolic markings because in this way our thinking is more general and clearer.* Therefore, we will call the train length l, and the unknown length of the bridge d. Travelling at a speed of v, during the bridge's vibration the train travelled the distance of $l + d$.



Therefore, according to the formula for uniform motion, the relation among the mentioned values is as follows:

$$l + d = v \cdot t$$

When we subtract l from both sides of the equation, the unknown value d will be expressed by means of the remaining known values:

$$d = vt - l = 20 \cdot 6 - 70 = 50 \text{ metres}$$

3. We cannot obtain s here according to the formula $s = v \cdot t = 6 \cdot 2 = 12$ metres. The formula $s = v \cdot t$ is valid, of course, only for uniform motion, and we are talking here about accelerated motion. Thus, the formula $s = v \cdot t$ cannot be applied here. If we assume that Romeo's speed was increasing uniformly, then we can use the formulas for uniform accelerated motion. It is known from physics that in such a situation the travelled distance s depends on the passed time t, speed v_0 at the beginning, and speed v at the end of the passed time t, according to the next formula:

$$s = \frac{v_0 + v}{2} t$$

Since in our case the speed at the beginning was zero, it follows that

$$s = \frac{0 + 6}{2} \cdot 2 = 6 \text{ metres}$$

□

How the Formulas are Applied

We often link the unknown value with the known ones by applying a formula (equality) which expresses the law of the given situation. In order to apply the formula correctly, we have to understand it. We have to know in which situations it is applicable and which quantities it links.

The *SageMath* command *solve* also helps us to express one quantity by means of others from the given equality. For instance, from the formula for the resistance of the connection in parallel $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ we can express the resistance R_2 by means of other resistances:

```
var('R,R1,R2')
show(solve(1/R==1/R1+1/R2,R2))
```

We will obtain

$$R_2 = -\frac{RR_1}{R - R_1}$$

Let us return to solving equations. The basic rule in solving equations is simple and understandable: *apply the same operation to both sides of the equation.* On this basis we can deduce other rules which are in certain situations simpler. Let us solve, for example, the equation $3x+6=15$:

$$3x+6=15 \quad |-6 \quad \rightarrow \quad 3x=15-6 \quad \rightarrow \quad 3x=9 \quad |:3$$
$$\rightarrow \quad x=\frac{9}{3} \quad \rightarrow \quad x=3$$

The total effect of subtracting the number 6 on both sides of the equation is that on the left side the adding of the number 6 is eliminated, and on the right side the subtracting of the number 6 has appeared. The total effect of division by the number 3 is that on the left side the multiplication by the number 3 is eliminated, and on the right side the division by the number 3 has appeared. Thus, you can convince yourself of the correctness of the following rules, and I believe you know them from school.

Moving Descriptions from One Side of the Equation to the Other

- The description which on the one side of the equation is added to the remaining part can be moved to the other side, where it is subtracted from that side, and vice versa.
- The description of a non-zero number which on the one side of the equation is multiplied with the remaining part can be moved to the other side using it for the division of that side, and vice versa.

Example 3.8.8. Let us solve the following equations by the rules of transposing:

1. $2x - 3 = 4x - 5$ 2. $\dfrac{3x}{4} = \dfrac{5}{6}$

Solution.

1. $2x - 3 = 4x - 5 \quad \rightarrow \quad 2x - 4x = -5 + 3 \quad \rightarrow \quad -2x = -2 \quad \rightarrow$
$x = \dfrac{-2}{-2} \quad \rightarrow \quad x = 1$

2. $\dfrac{3x}{4} = \dfrac{5}{6} \quad \rightarrow \quad x = \dfrac{5 \cdot 4}{6 \cdot 3} \quad \rightarrow \quad x = \dfrac{10}{9}$ □

When we link the known and unknown quantities, it may happen that we have several unknown quantities and several pieces of information about them. Then the translation is not one equation, but **a system of equations**, several equations with several unknowns. **The solution of a system of equations** (of a system of conditions) with, for example, two unknowns (with two variables) is a pair of numbers which, when taken in order for the values of variables, fulfil (satisfy) all the equations (conditions) of the system. For instance, for the system of equations

$$\begin{aligned} x + 2y &= 4 \\ x - y &= 1 \end{aligned}$$

the pair (1,2) is not the solution because when instead of x we put the number 1 and instead of y the number 2 we will obtain false statements:

$$\begin{aligned} 1 + 2 \cdot 2 &= 4 \\ 1 - 2 &= 1 \end{aligned}$$

In the same way, the pair (0,2) is not the solution either, since, although it satisfies the first equation, it does not satisfy the second one:

$$\begin{aligned} 0 + 2 \cdot 2 &= 4 \\ 0 - 2 &= 1 \end{aligned}$$

However, the pair (2,1) is the solution since it satisfies both conditions:

$$2 + 2 \cdot 1 = 4$$
$$2 - 1 = 1$$

In order to make it clearer which number in the pair we have replaced with which variable, we also say that the solution of the system is the pair $x = 2$ and $y = 1$.

Example 3.8.9. In a forest by the road, the cruel Istrian Luciano's gang (Istria is a very peaceful part of Croatia) were relaxing. Along the path the gang of an even more cruel Istrian, Jakob, came along. Respecting the unspoken agreement on sustainable development they did not have a fight but rather a friendly discussion. Luciano remarked "*If you gave us one of your men, we would have the same number of men as you.*" Not wanting to lag behind in this intellectual conversation Jakob answered "*If you gave us one man, we would have twice as many men as you.*" They did not know that Inspector Clouseau was sitting in the cavity of a tree, and noting everything down carefully. But in the process, he clumsily elbowed a wasps' nest in the tree. The wasps stung him so many times that he could not in any way conclude how many members each gang had. Let us help the clumsy inspector.

Solution. Let us designate the number of people in Luciano's group with L and in Jakob's group with J. Then the given information can be transformed into the following equations:

$$L + 1 = J - 1$$
$$(L - 1) \cdot 2 = J + 1$$

We have obtained two pieces of information about two unknowns. The permitted steps in discovering the unknowns are the same as in the case of one equation, but now we apply them to both equations. The strategy is as follows. We will use one equation in order to describe one unknown by means of the other. For instance, L can be expressed from the first equation by means of J:

$$L = J - 2$$

Thus, we have used the first piece of information to describe L by means of J. In this way we have reduced the problem to searching for J. When we determine J, from the above description we will easily calculate L. In order to find J, we will use the second item of information. True, it talks also about L and about J, but now we know how to express L by means of J. When in the second piece of information L is replaced by its description by means of J, $L = J - 2$, it will talk only about J:

$$(J - 2 - 1) \cdot 2 = J + 1$$

Thus, we have reduced the problem of solving two equations with two unknowns to the problem of solving one equation with one unknown. Formally, we achieved this by replacing L with the description by means of J, and this method of solving the system is called the **substitution method**. Now we solve the equation with one unknown. We easily get that $J = 7$. Knowing J, from the description of L by means of J we will calculate also L: $L = J - 2 = 7 - 2 = 5$.

Dear Inspector Clouseau, Luciano's gang has 5 members, and Jakob's has 7. □

Method of Substitution

The method of substitution is the following procedure for solving a system of equations:

1. We use one equation to express one unknown by means of the remaining ones.
2. In all the other equations we replace this unknown with the obtained expression.
3. We will get fewer equations with fewer unknowns.
4. With the obtained equations we repeat the procedure until we get one equation with one unknown.
5. By solving this equation we will obtain not only its unknown, but, going back to the previous equations, we will obtain all the unknowns.

The method of substitution is the most efficient method of solving a system of equations, but it is not omnipotent. It may occur that in no equation can one unknown be substituted by means of another one, or on the other hand that we obtain an excessively complex description. Then an attempt is made to solve the system in another way – **by combination** of more complex equations into simpler ones. I will illustrate this on the following system of equations:

$$4x + 7y = 6$$
$$6x + 11y = 8$$

This system could be solved by the substitution method, but the procedure would get a little bit complicated (try it). Instead, we will combine the equations into a simpler one. Combining is founded on the logical property of equality that the "same combination of equals yields equal":

$$L_1 = R_1,\ L_2 = R_2 \quad \rightarrow \quad C(L_1, L_2) = C(R_1, R_2)$$

where C is any operation (combination) of two numbers. For instance, if we added the left sides of the equations and the right sides of the equations (then we say that we have added the equations) we would get the equation

$$\begin{array}{rcl|l} 4x + 7y & = & 6 & \\ 6x + 11y & = & 8 & + \\ \hline 4x + 7y + 6x + 11y & = & 6 + 8 & \end{array}$$

The obtained equation is no simpler than the initial ones. However, if there were opposite coefficients of one unknown, we would obtain a simpler equation, since this unknown would disappear from the equation. Therefore, we will first multiply the equations by appropriate numbers so that we get opposite coefficients of one unknown. For instance, in order to obtain opposite coefficients of the unknown x, we will multiply the first equation by -3 and the second by 2 (we will get $\pm$ 12 – the smallest common multiple of the existing coefficients 4 and 6):

$$\begin{array}{rcll} 4x+7y & = & 6 & |\cdot(-3) \\ 6x+11y & = & 8 & |\cdot 2 \end{array}$$

Now that we have obtained opposite coefficients of the unknown, we will add the equations:

$$\begin{array}{rcll} -12x-21y & = & -18 & \\ 12x+22y & = & 16 & \big|+ \\ \hline -12x+12x-21y+22y & = & -18+16 & \end{array}$$

We have obtained a simple equation in which one unknown *has disappeared*: $y=-2$. If we introduce this value for y, for instance into the first equation, we will get the corresponding x:

$$4x+7\cdot(-2)=6 \quad \rightarrow \quad 4x=6+14 \quad \rightarrow \quad x=5$$

Thus, the solution of the system is the pair $(5,-2)$. This method of solving by combination is called the **method of opposite coefficients**. Naturally, there are also other ways that combinations can be used.

Method of Opposite Coefficients

The system of two linear equations with two unknowns (the equations of the form $ax+by=c$) is best solved by using the **method of opposite coefficients**:

1. We multiply each equation with an appropriate number to obtain opposite coefficients of one of the unknowns.

2. By adding the equations we obtain one equation with one unknown, by which we determine this unknown.

3. We introduce the obtained value of the unknown into one of the initial equations in order to get the respective value of the second unknown.

Example 3.8.10. Let us solve the following systems:

1. $$\begin{aligned} -3x+5y &= 8 \\ x+7y &= 6 \end{aligned}$$

2. $$\begin{aligned} 2x-4y &= 1 \\ 4x-8y &= 2 \end{aligned}$$

3. $$\begin{aligned} -3x+2y &= 7 \\ -6x+4y &= 7 \end{aligned}$$

Solution.

1. We will multiply the second equation by 3 in order to get the opposite coefficients of the unknown x:

$$\begin{array}{rcll} -3x+5y & = & 8 & \\ 3x+21y & = & 18 & \Big|+ \\ \hline 26y & = & 26 & \\ \rightarrow y & = & 1 & \end{array}$$

We insert the obtained value for y into the second equation:

$$x+7\cdot 1=6 \quad \rightarrow \quad x=-1$$

The solution is the pair $(-1,1)$.

2. We will multiply the first equation by -2 in order to get the opposite coefficients of the unknown x:

$$\begin{array}{rcll} -4x+8y & = & -2 & \\ 4x-8y & = & 2 & \Big|+ \\ \hline 0 & = & 0 & \end{array}$$

When we add the equations, not only does one unknown disappear, but the second one disappears as well and we obtain something that already know: that $0 = 0$. The fact that the equations are cancelled by addition means that they are very similar. Indeed, if we were to multiply the first equation by 2 instead of -2, we would obtain precisely the second equation:

$$2x - 4y = 1 / \cdot 2 \quad \rightarrow \quad 4x - 8y = 2$$

This means that the two equations (conditions) are equivalent (have the same solutions), so that one is redundant (it does not provide new information). We can forget it, since its solutions are identical to the solutions of the remaining equation. Thus the solutions are all pairs of numbers that satisfy this condition. If we express x by means of y

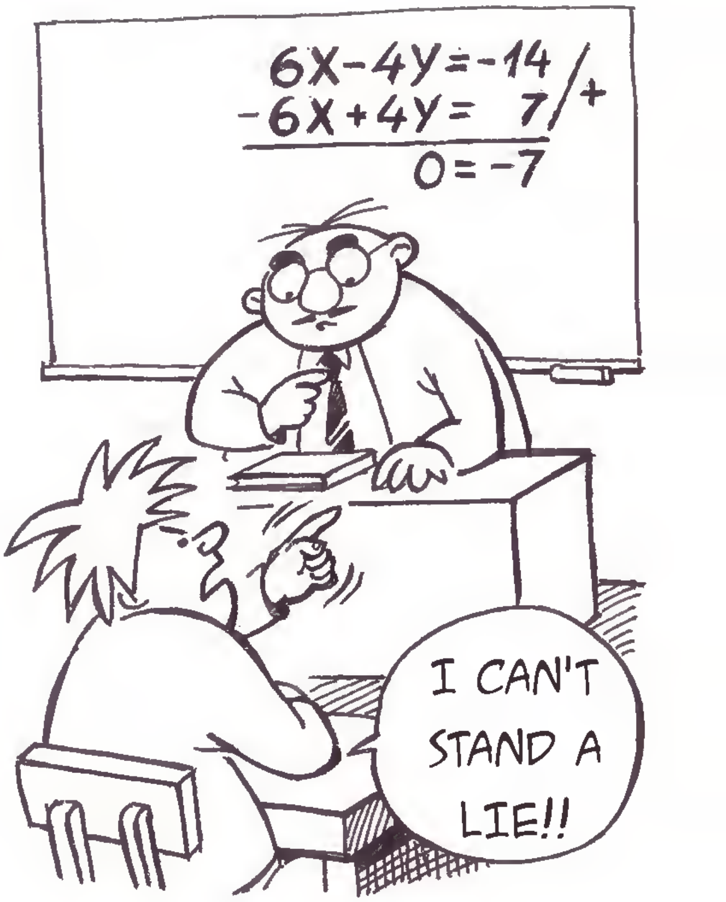
$$x = \frac{1}{2} + 2y$$

we can see that whichever number we take instead of y (e.g. $y = 1$) we will get an adequate value for x (for $y = 1$, $x = \frac{5}{2}$). Thus this system has infinite solutions, all pairs of numbers of the form $(\frac{1}{2} + 2y, y)$ where y is an arbitrary number.

3. We will multiply the upper equation by -2 in order to obtain the opposite coefficients of the unknown x:

$$\begin{array}{rcl|l} 6x - 4y & = & -14 & \\ -6x + 4y & = & 7 & + \\ \hline 0 & = & -7 & \end{array}$$

Chasing the unknowns, we have obtained a falsehood! However, this can be easily interpreted. Indeed, *transforming from one set of equations to another by permitted steps is in fact, as we have already pointed out, correct conclusion-making.* It shows us that if for some x and y (that we are searching for) the initial equalities are valid, then the final ones are valid as well. If, however, the final equality is false, it means that the initial set of equalities for any x and y fail to be fulfilled either. We also say that the given set of equations (conditions) is unsatisfiable or that it is contradictory. Thus, this system of equations has no solution. □



The command *solve* in the software *SageMath* also helps us in solving systems of equations. Using it, we will solve, for example, the first system of equations from the previous example, with the following command:

```
var('y')
solve([-3*x+5*y==8,x+7*y==6],x,y)
```

We will get

```
[[x == -1, y == 1]]
```

Example 3.8.11. A messenger pigeon started from town A towards town B flying at a uniform speed of 20 km/h. At the same time (from town B towards town A) there started a messenger dove flying at a speed of 15 km/h. If the towns are at a distance of 7 km, at what point and when will the pigeon meet the dove?

Solution. Let us call the known speeds of the pigeon and the dove $v_1 = 20$ km/h and $v_2 = 15$ km/h, and the distance of the towns $d = 7$ km. The unknown paths that the pigeon and the dove had flown before their encounter will be called s_1 and s_2, and the time until the encounter t. We will apply the formulas for uniform motion and the relationship between the distances to connect the unknown values with the known ones:

$$s_1 = v_1 t \qquad s_2 = v_2 t \qquad d = s_1 + s_2$$

Since the first two equations describe s_1 and s_2 by means of t, we will replace with these descriptions s_1 and s_2 in the third equation and get one equation with one unknown:

$$d = v_1 t + v_2 t \quad \to \quad d = (v_1 + v_2)t \quad \to$$

$$t = \frac{d}{v_1 + v_2} = \frac{7\,\mathrm{km}}{20\,\mathrm{km/h} + 15\,\mathrm{km/h}} = \frac{1}{5}\,\mathrm{h} = 12\,\mathrm{min}$$

Thus, the time until encounter is 12 minutes, and the covered paths of the pigeon and the dove are

$$s_1 = v_1 t = 20\,\frac{km}{h} \cdot \frac{1}{5} h = 4\,km \qquad s_2 = v_2 t = 15\,\frac{km}{h} \cdot \frac{1}{5}\,h = 3\,km$$

The command *solve* in the software *SageMath* helps us in solving the system of equalities. Thus, from the system of equalities from the previous example we will express the unknown values s_1, s_2 and t by means of the known values v_1, v_2 and d by the following command:

```
var('s1,s2,t,v1,v2,d')
show(solve([s1==v1*t,s2==v2*t,d==s1+s2],s1,s2,t))
```

We will get

$$[[s_1 = \frac{dv_1}{v_1 + v_2},\ s_2 = \frac{dv_2}{v_1 + v_2},\ t = \frac{d}{v_1 + v_2}]]$$

Let us dwell a little more on the application of equations in solving "actual" problems. A problem described in natural language was translated into mathematical language. In this translation we named all the values we did not know. Every item of information about unknown values was translated into one equation. In this linking of known and unknown values we also used the laws (which are equalities) that "cover" this situation. The translation was a system of equations, a well-defined problem in mathematical language for which we have simple means of solutions that enable us, unlike in natural language, to efficiently find the solutions. However, in the mathematical part we only solved the mathematical problem (found the solutions to the system of equations). This does not necessarily have to be the solution of the actual problem, since it is possible that we made a mistake in translation, used false laws or maybe overlooked some important characteristics of the situation. However, this is no longer a matter of mathematics but rather of knowing the area to which mathematics has been applied. This will be discussed in more detail in *Circle 2*.

The application of equations illustrates a typical method of applying mathematics:

Typical Application of Mathematics

The actual problem, taking into account the laws of the domain of the problem, with some simplifications and by means of the meaning of mathematical notions, is translated from natural language into mathematical language. The translation is a mathematical object (in our case this was a system of equations). We also say that we have obtained a mathematical model of the problem. In mathematical language we usually have efficient formal methods that provide a solution. However, they give the solution to the mathematical problem. This solution needs to be interpreted so as to see whether it is also the solution to the actual problem. It is possible that the translation of the actual problem into the mathematical problem was bad. However, the translation skill is no longer simply a matter of knowing mathematics, but also of knowing the context in which the problem has occurred.

Usually, several laws describe a certain situation, and not just one. We use them in order to obtain new laws or in order to combine in the given situation several unknown values with the known ones. For a concrete example we can take the situation of uniform accelerated motion.

Example 3.8.12. Uniformly accelerated motion is motion under the influence of constant force. As, for instance, when you accelerate uniformly the speed of a car (and neglect the variation in air resistance at the change of speed). Here, the speed v of the body rises uniformly according to the law $v = v_0 + at$ and the travelled path s increases according to law $s = v_0 t + \frac{1}{2}at^2$, where v_0 is the initial speed, a is acceleration of the body and t is the time passed.

1. By eliminating t let us find the connection between the remaining quantities.
2. By eliminating a let us find the connection between the remaining quantities.

Solution.

1. Let us express from the equation $v = v_0 + at$ the time t using the remaining values, $t = \dfrac{v - v_0}{a}$, and let us eliminate it from the equation $s = v_0 t + \frac{1}{2}at^2$:

$$s = v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2$$

Applying the rules for equalities, we will get

$$v^2 = {v_0}^2 + 2as$$

2. Let us express from the equation $v = v_0 + at$ the acceleration a using the remaining values, $a = \dfrac{v - v_0}{t}$, and let us eliminate it from the equation $s = v_0 t + \frac{1}{2}at^2$:

$$s = v_0 t + \frac{1}{2}\frac{v - v_0}{t}t^2$$

Applying the rules for equalities, we will get

$$s = \frac{v_0 + v}{2} \cdot t$$

Thus, from certain laws that describe a situation, we have derived new laws for that situation! □

The last example illustrates the most important property of the language of equalities:

The Language of Equalities and Laws of Nature

By measuring, we assign numbers to physical phenomena. These numbers characterise the phenomena. Through these assignments the laws of phenomena often occur as equalities between measured values. Having efficient tools for work with equalities (simplifying each side of the equality and applying the same operation to both sides of the equality) we obtain from one set of equalities some others. Thus we discover new laws in the considered phenomena.

In the *SageMathTutorial* on the web site of the book SageMath is described in more detail how the equations in the software *SageMath* are solved.

You can find more about application of equations in solving "real" problems at `https://en.wikipedia.org/wiki/Word_problem_(mathematics_education)`