Introduction to Numbers

Number rules the universe.

Pythagoras, 6th century BCE philosopher

Numbers are the free creation of the human mind.

Richard Dedekind, one of the creators of modern mathematics (1831–1916)

At the very beginning I must say that numbers are strange in one aspect. Where are they? Biology deals mainly with plants and animals. We can see precisely what it studies. History deals with humans and events that are not here any more, but whose existence is easy to imagine. Physics deals, for instance, with atoms. We cannot see them, but we can understand where they are and why we do not see them. But, what is it with numbers? And what about other mathematical objects? Some of you will probably have got acquainted with vectors and matrices, maybe with even more complex mathematical objects, such as differential forms on manifolds or Hermitian operators on infinite-dimensional complex Hilbert spaces. But do they really exist or are they as fictional as creatures from fairy tales, like fairies and dwarfs, trolls and gremlins? Isn't it a little bit odd, after you have spent so much time calculating with numbers, that, probably, nobody has ever told you whether they exist at all, never mind where they actually are. This is one of those "embarrassing" questions that people try to avoid. Because the answer is simple and disturbing: *it is not known whether numbers do exist!* For thousands of years, mathematicians and those who helped (hindered?) them have not found an answer. Or, rather, they have found several answers and are still arguing about which one is the right one. You can see the current situation of the discussion at the website `https://en.wikipedia.org/wiki/Philosophy_of_mathematics`. But be careful not to get too bound up in it!

The quotations at the beginning of the chapter express two extreme attitudes. The ancient Greek Pythagoras was convinced that numbers not only exist but are the basis of the entire world: the basic laws of nature are mathematical laws of numbers and they determine everything – from the way the stars will move, to with whom you will fall in love. Maybe it is not clear how the wise Pythagoras could say something so unwise, but history has shown that this blind faith of his in numbers has been very fruitful. However, if Pythagoras was right, where in the world are these numbers and how do we reveal the truths about them? More than two thousand years later, Dedekind's thinking took him in a completely different direction. He considered numbers to be a free human creation. But if we did invent numbers, how do they help us manage in the world?

The contemporary viewpoint is closer to Dedekind: numbers and other mathematical objects do not actually exist; what is more, they resemble the creatures from fairy tales. A good fairy tale touches us deeply and teaches us through its world of imagined characters and events that, for instance, love is more valuable than money or we should not judge someone by their name. Similarly, a good mathematical fairy tale attracts attention by its inner harmony and provides us with a powerful tool to solve problems. *The world of numbers is an imagined world, not exactly arbitrarily imagined, but rather imagined for a particular purpose, its use being to count and measure*. For example, each natural number in the process of counting has a precisely defined role. And it is only this role that is important, and not out of what it is made. The same as how chess pieces may be made of wood or plastic, or even be drawings on the computer screen, but it is their role in the game that is important. A piece is a king, not because it was made of iron or beech, but because it can move one square at a time , and if it is checkmated by another piece, the game is lost. It is the same with numbers. The number 3 can be imagined as three small lines on paper or as three knots on a rope. However, the only important thing is its role in the imagined structure (world) of natural numbers – that it is a number following number 2 and preceding number 4. This is also my view of numbers, as a world imagined with a certain purpose. Such a view of mathematical objects has a certain advantage, since it leaves us the freedom to imagine any worlds, regardless of whether they do actually exist or not. This view of mathematics also proved historically significant. It was giving up on the older philosophy of mathematics (according to which mathematics describes eternal truths about eternal objects) and accepting the new philosophy, in which mathematical worlds are free products of our mind, that liberated human mathematical powers in the 19th century and led to the true flourishing of modern mathematics that still continues today.

Mathematical Worlds

Mathematical worlds are imagined worlds with the purpose of being powerful tools in studying and shaping the world, or to put it more concretely, in solving problems.

Although I will stick to Dedekind's approach to mathematics, you do not have to agree with my opinion, just as many other mathematicians and philosophers have not. If you find it more acceptable to think that numbers do exist in a certain way and that we did not invent their properties but just discovered them, feel free to imagine them like that. Once we have agreed on their properties and method of functioning, this issue will have no further impact on our work. Mathematics is such a powerful and lively human activity that it functions perfectly well even without a final answer to the question of whether mathematical objects exist or not. *When the world of numbers, real or imagined, started to be connected with the real world in the processes of counting and measuring, the human community came to unprecedented knowledge and the possibilities of control*. This started with the ancient Greeks: first of all with Pythagoras. He was probably the first who understood what immense power is given to us by numbers, so maybe it was in this enthusiasm that he exclaimed that numbers manage the world. However, the first to implement this power in practice were the creators of modern science in 17th century: Galileo, Kepler and Newton. They and their successors, by counting, measuring, and thinking, provided the description of nature which is the basis of our modern-day understanding of universe and the technical advances that facilitate (or aggravate) our life.

If, by any chance, Pythagoras and Dedekind were to meet today, although they would not be able to agree on the issue of the real nature of numbers, they would most certainly agree that *the numbers are the basis of our description of the world*. Without numbers we can state that there is a lot or a little of something, but with them we can resolve this much more precisely. For instance, there are exactly 17 books on my shelf and by counting them I can easily find out whether there any are missing. Also, in human language there are only a few words for the common colours (blue, yellow, dark red, etc.). However, with the use of numbers we can say very precisely that, for example, the yellow colour of helium is not just any kind of yellow but rather a yellow of a particularly defined wavelength of 5.88×10^{-7} metres. Not only can we use measuring to determine something more precisely, but also regularities in nature can be expressed precisely by the relations between numbers. For instance, when a steel bar is heated, its elongation Δl is proportional to the existing length of the bar l and the change in temperature Δt ($°C$ denotes the degree Celsius – a unit for temperature):

$$\Delta l = 16 \cdot 10^{-6}\ {}^{\circ}C^{-1} \cdot l \cdot \Delta t$$

This law should be considered in the case of all steel structures, ranging from railway lines to bridges. It can be used to calculate exactly the elongation of the steel elements, and hence design the structures so that such elongation does not disturb their stability or functionality.

These are some almost randomly selected examples that show how numbers are important and why good knowledge of them is necessary. They also illustrate the essential characteristic of all good mathematical ideas – *they are a powerful tool for investigating and designing the world*.

Numbers are just one of the realisations of the general idea of measurement. In this first circle we will get to know also a different realisation of this idea, the coordinate system. There are also some other very important realisations of the idea of measurement which I will only mention in the *Inspiring End* of the book, and some of them will be studied in subsequent Circles.

Measure what is measurable, and make measurable what is not so.

Galileo Galilei, scientist (1564–1642)