An Elementary System of Axioms for Euclidean Geometry Based on Symmetry Principles

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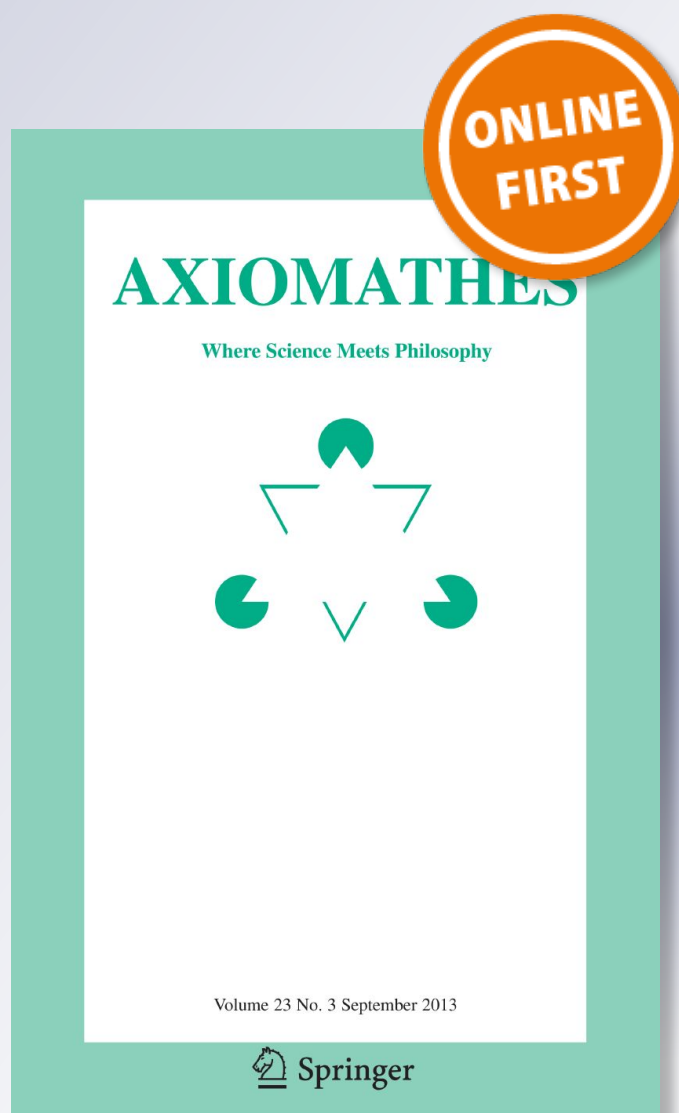
Boris Čulina[1]

Abstract In this article I develop an elementary system of axioms for Euclidean geometry. On one hand, the system is based on the symmetry principles which express our a priori ignorant approach to space: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and all units of length we use to create geometric figures are the same to us (the scale invariance of space). On the other hand, through the process of algebraic simplification, this system of axioms directly provides the Weyl's system of axioms for Euclidean geometry. The system of axioms, together with its a priori interpretation, offers new views to philosophy and pedagogy of mathematics: (1) it supports the thesis that Euclidean geometry is a priori, (2) it supports the thesis that in modern mathematics the Weyl's system of axioms is dominant to the Euclid's system because it reflects the a priori underlying symmetries, (3) it gives a new and promising approach to learn geometry which, through the Weyl's system of axioms, leads from the essential geometric symmetry principles of the mathematical nature directly to modern mathematics.

Keywords Symmetry · Euclidean geometry · Axioms · Weyl's axioms · Philosophy of geometry · Pedagogy of geometry

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1 Introduction

The connection of Euclidean geometry with symmetries has a long history. I will be concerned here with the three symmetry principles: (1) the homogeneity of space: all places are the same, (2) the isotropy of space: all directions are the same, (3) the scale invariance of space: all units of length we use to create geometric figures are the same. In 17th century John Wallis proved, assuming other Euclid's postulates, that the scale invariance principle "For every figure there exists similar figure of arbitrary magnitude." is equivalent to the Euclid's fifth postulate (Wallis 1695–1699). Wallis considered his postulate to be more convincing than Euclid's fifth postulate. Tracing back to the famous Riemann lecture at Göttingen in 1854 "Über die Hypothesen welche der Geometrie zu Grunde liegen" (Riemann 1867), it is well known that among all Riemann manifolds Euclidean geometry is characterized by the three symmetry principles. However, this characterisation is not an elementary one because it presupposes the whole machinery of Riemann manifolds. The descriptions based on Klein's program of characterizing geometries by their principal groups of transformations (Klein 1872) also have non elementary character. As I am aware, there is no an elementary description (a description in terms of intuitive relations between points) of Euclidean geometry that is based on the three symmetry principles. Here I develop a system of axioms that provides such an elementary description.

The importance and validity of the three symmetry principles has been recognized a long time ago. I offer a new interpretation of these principles. Clifford (1873, 1885) considers the three symmetry principles as the most essential geometrical assumptions. He considers that the principles are based on observations of the real space. Von Helmholtz (1868) has the same opinion for the first two symmetry principles which he unifies in his principle of the free mobility of rigid bodies. Henri Poincaré, in his analysis of the real space (Poincare 1902), comes to the conclusion that the first two symmetry principles are the most essential properties of the so called geometric space which for him is not the real space but a "conventional space"—the most convenient description of the real space. An interesting explanation of the validity of the three symmetry principles comes from Delboeuf (1860). He considers what remains when we ignore all differences of things caused by their movements and mutual interactions. According to Delboeuf, in the ultimate abstraction from all diversities of real things we gain the homogeneous, isotropic, and scale invariant space - the true geometric space which is Euclidean and which is different from the real space. My view on the principles is a different one: they are not a posteriori, the result of analysing the real space, but they are a priori. However, in my interpretation they are not a priori in Kant's sense, as an a priori form of rational cognition, but in a sense of our a priori ignorant approach to space: all places are the same to us (the homogeneity of space), all directions are the same to us (the isotropy of space) and all units of length we use for constructions in space are the same to us (the scale invariance of space). Because the system of axioms I develop here is based on the three symmetry principles and because it is equivalent to other systems of axioms for Euclidean geometry, under my a priori interpretation of the principles, the system supports the thesis that

Euclidean geometry is a priori, in the same way as number systems are a priori, the result of modelling, not the world, but our activities in the world. This conclusion could satisfy Gauss who expressed his dissatisfaction with the epistemic status of Euclidean geometry in a letter to Olbers (Gauss 1900): "I am ever more convinced that the necessity of our geometry cannot be proved, at least not by, and not for, our human understanding. Maybe in another life we shall attain insights into the essence of space which are now beyond our reach. Until then we should class geometry not with arithmetic, which stands purely a priori, but, say, with mechanics."

The system of axioms I develop here directly provides, through the process of algebraic simplification, the Weyl's system of axioms for Euclidean geometry (Weyl 1918).[1] Thus, the system of axioms I develop here, together with my a priori interpretation of the system, offers an explanation of the fact that in modern mathematics the Weyl's system of axioms is dominant to the Euclid's system. Although the Euclid's system of axioms for Euclidean geometry is thought in school, the Weyl's system of axioms is used in modern mathematics, physics and engineering. Only through the Weyl's system of axioms do we find the Euclidean structure in complex mathematical structures, and this enables us to make our reasoning about them more visual and efficient. Today, the Weyl's system of axioms is one of the essential synthesizing tools of modern mathematics while Euclid's system is of a secondary importance. My thesis is that this happens because the original Euclidean system of axioms reflects a posteriori intuition (hence physical intuition) about Euclidean geometry while the Weyl's system of axioms reflects a priori intuition (hence mathematical intuition) about the underlying symmetries.

The system of axioms I develop here also gives new and promising approach to learn Euclidean geometry because (1) it gives an elementary and essential description of Euclidean geometry, (2) it gives, through the process of algebraic simplification, the Weyl's system of axioms for Euclidean geometry which is essential for modern mathematics. Therefore, it opens the possibility to learn geometry in a way which leads from the essential geometric symmetry principles (and these are of the mathematical nature) directly to modern mathematics.

I gradually introduce a system of axioms about points labelled with letter "A", which have an immediate justification in intuitive ideas about relations between points and in the three symmetry principles supported by an idea of continuity of space. Then, I deduce from them The Weyl's system of axioms about points and vectors labelled with letter "W", which are indeed equivalent to A—axioms. The complete list of axioms is displayed in the "Appendix".

The primitive terms of the system of axioms are: (1) equivalence of pairs of points (arrows), (2) multiplication of a pair of points by a real number and (3) distance between points. The multiplication could be avoided. However, the procedure to define the multiplication is somewhat lengthy and I prefer to introduce the multiplication as a new primitive term. Also, it is more simple to introduce the distance function (to add an arbitrary unit of measurement) as a new primitive than to introduce congruence between pairs of points as a new primitive term and define the distance function relative to the choice of a unit of measurement.

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[1] brief description of the Weyl's system is on pages 16 and 30.

2 Equivalence of Pairs of Points

Geometrical space S will be modelled as a non empty set of objects termed **points**. The basic geometrical relation is the position of one point relative to another (not necessarily different) point. That the position of a point B relative to a point A is the same as the position of a point B' relative to a point A' we will denote $AB \sim A'B'$ and we will say that pairs or arrows AB and $A'B'$ are **equivalent**. This is the first primitive term of our system. It expresses a basic intuitive idea about relation of two points. The idea itself *to be in the same relative position* implies that it is a relation of equivalence. This is the content of the first axiom.

Axiom A1 $\sim$ *is an equivalence relation.*

In more detail, it means:

Axiom A1.1 $AB \sim AB$ (*reflexivity*)

Axiom A1.2 $AB \sim A'B' \rightarrow A'B' \sim AB$ (*symmetry*)

Axiom A1.3 $AB \sim A'B' \wedge A'B' \sim A''B'' \rightarrow AB \sim A''B''$ (*transitivity*)

Because of an elementary character of these properties of $\sim$, usually I will not mention them in proofs of theorems.

Concerning a fixed point A we can easily describe the equivalance relation $\sim$: by the very idea of the relative position of points, different points have different relative positions to A:

Axiom A2 $AB \sim AC \rightarrow B = C$.

Fundamental operations with arrows are to invert an arrow and to add an arrow to another arrow. The definitions follow:

inverting arrow $AB \mapsto -AB = BA$

addition of arrows $AB, BC \mapsto AB + BC = AC$

Because of Axiom A2 we can extend addition of arrows:

generalized addition of arrows $AB + CD = AB + BX$, where $BX \sim CD$, under the condition that there is such a point X.

By the homogeneity principle, the operations are invariant under the equivalence of arrows:

Axiom A3.1 $AB \sim A'B' \rightarrow BA \sim B'A'$.

Axiom A3.2 $AB \sim A'B' \wedge BC \sim B'C' \rightarrow AC \sim A'C'$

Until now, we know only that AB is equivalent to itself (reflexivity of $\sim$) and to no other arrow from the point A (Axiom A2). All other axioms are conditional statements. It remains to describe the equivalence of arrows originating from different points.

3 Multiplication of an Arrow by a Number

In the next section I will sketch how can we define the multiplication of an arrow by a number and establish basic properties of the operation. Although such approach is conceptually more satisfactory, it is technically very lengthy. Instead, I will directly postulate the properties of multiplication of arrows by real numbers, without losing a clear basis of the three symmetry principles. I introduce **multiplication of an arrow by a real number** as a new primitive operation based on an idea of stretching arrows and of an idea of iterative addition of the same arrow (numbers will be labelled with letters from the Greek alphabet):

$$\cdot : \mathbb{R} \times S^2 \to S^2 \quad \lambda, A, B \mapsto \lambda \cdot AB$$

Sometimes, since it is a common convention, we will not write the multiplication sign at all.

The very idea of the multiplication as stretching arrows is formulated in the next axiom:

Axiom A4 $\forall \lambda, A, B \; \exists C \; \lambda \cdot AB = AC$.

By the homogeneity principle, multiplication of an arrow by a number is invariant under the equivalence of arrows:

Axiom A5 $AB \sim CD \; \to \; \lambda AB \sim \lambda CD$.

For a point C such that $AC = \lambda \cdot AB$ we will say that it is **along** AB. Also, for arrow AC we will say that it is **along** AB.

The very idea of the multiplication as addition of the same arrow leads to the next axiom:

Axiom A6.1 $1 \cdot AB = AB$.

By the homogeneity principle, we can translate any arrow along AB to any point along AB. So, we can add such arrows. Specially, we can add $\lambda \cdot AB$ and $\mu \cdot AB$ and the result will be $\lambda \cdot AB + \mu \cdot AB = v \cdot AB$ for some number v. Moreover, by the very idea of the multiplication as iterative addition of the same arrow $v = \lambda + \mu$. This is the content of the next axiom:

Axiom A6.2 $\lambda \cdot AB + \mu \cdot AB = (\lambda + \mu) \cdot AB$.

Let's note that with this equation we postulate also that the left side of the equation is defined.

If we stretch an arrow along AB the result will be an arrow along AB, too. So, $\lambda \cdot (\mu \cdot AB) = v \cdot AB$, for some number v. Moreover, by the very idea of the multiplication as iterative addition of the same (stretched) arrow $v = \lambda \cdot \mu$. This is the content of the next axiom:

Axiom A6.3 $\lambda \cdot (\mu \cdot AB) = (\lambda \cdot \mu) \cdot AB$.

Let's note that with this equation we postulate also that $\lambda \cdot (\mu \cdot AB)$ is along AB. Later, we will need to translate an arrow on its both sides:

Corollary A′4 $\forall A, B \; \exists D \; AB \sim BD$.

Proof $2 \cdot AB = (1 + 1) \cdot AB = (\text{by A6.2})1 \cdot AB + 1 \cdot AB = (\text{by A6.1})AB + AB$. By the meaning of addition there is the point D such that $AB \sim BD$.   □

Corollary C1 $\forall A, B \; \exists C \; CA \sim AB$.

Proof   Applying A'4 on BA we can find the point C such that $BA \sim AC$. Because inverting arrows is invariant under $\sim$ (A3.1) it follows that $AB \sim CA$.   □

The last axiom expresses the scale invariance principle.

Axiom A7   (*the scale invariance axiom*) *If $AC = \lambda \cdot AB$ and $AC' = \lambda \cdot AB'$ then $CC' \sim \lambda \cdot BB'$* (Fig. 1).

Corollary C2 $\lambda \cdot (AB + BB') = \lambda \cdot AB + \lambda \cdot BB'$.

Proof   By A4 there are points C and C' such that $\lambda \cdot AB = AC$ and $\lambda \cdot AB' = AC'$. Then, by Axiom A7, $CC' \sim \lambda \cdot BB'$. Now, we calculate: $\lambda \cdot AB + \lambda \cdot BB' = AC + CC' = AC' = \lambda \cdot AB' = \lambda \cdot (AB + BB')$.   □

Of the special interest is a somewhat modified special case of the scale invariance axiom, for $\lambda = 2$:

Theorem A′5   (*the elementary scale invariance law*) $AB \sim BC$ *and* $AB' \sim B'C' \;\rightarrow\; \exists P \; CP \sim PC' \sim BB'$ (Fig. 2).

Proof   Suppose that $AB \sim BC$ and $AB' \sim B'C'$. Then $AC = AB + BC \sim AB + AB = (\text{by A6.1})1 \cdot AB + 1 \cdot AB = (\text{by A6.2})2 \cdot AB$. In the same way we prove that $AC' = 2 \cdot AB'$. Then, by A7, $CC' \sim 2 \cdot BB'$. Multiplying this relation with $\frac{1}{2}$, by A5 and A6.3, it follows that $BB' \sim \frac{1}{2} \cdot CC'$. By A4, there is the point P such that $\frac{1}{2} \cdot CC' = CP$. Therefore, $CP \sim PC' \sim BB'$.   □

This theorem has two important consequences.

Theorem T3   (*the unique translation of arrows law*) $\forall A, B, A' \; \exists! B' \; AB \sim A'B'$.

Proof   We will accomodate notation to the elementary scale invariance law. We will prove that for all B, B', C there exists the unique point P such that $BB' \sim CP$. By C1 there is the point A such that $AB \sim BC$. By A'4 there is the point C' such that $AB' \sim B'C'$. By A'5 there is the point P such that $BB' \sim CP$. The point P is a unique such point. If there were another point P' such that $BB' \sim CP'$, then by symmetry (A1.2) and transitivity (A1.3) of $\sim$, we infer $CP' \sim CP$. From this statement, by A2, it follows that $P' = P$.   □

The unique translation of arrows law enables us to add arbitrary arrows, without any condition, as we have done before.

$$AB + CD = AB + BX, \quad \text{where } BX \sim CD$$

Theorem T4   (*the parallelogram law*) $AB \sim A'B' \;\rightarrow\; AA' \sim BB'$ (Fig. 3).
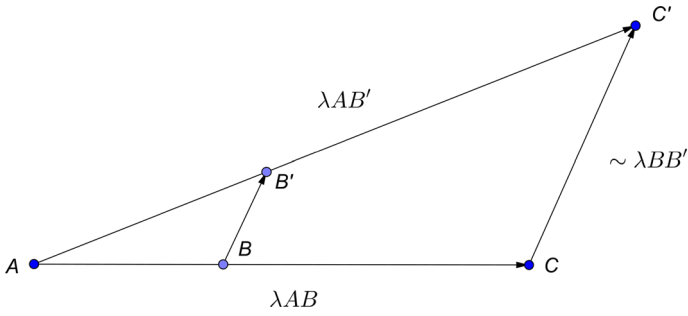
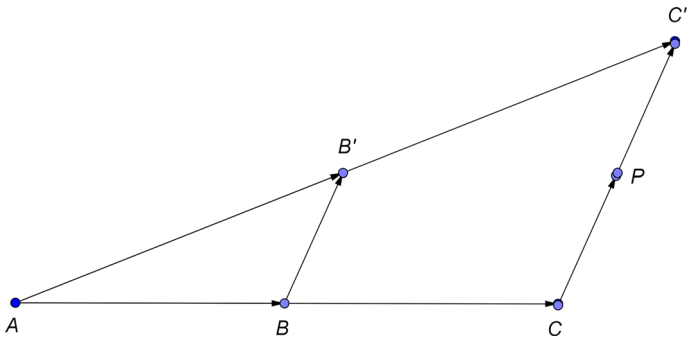**Fig. 1** Illustration for the scale invariance axiom

Fig. 2 Illustration for the elementary scale invariance law

Proof   We will accomodate notation to the elementary scale invariance axiom. We will prove that if $BB' \sim CP$ then $BC \sim B'P$. By C1 there is the point A such that $AB \sim BC$. By A'4 there is the point C' such that $AB' \sim B'C'$. By A'5 there is the point P' such that $CP' \sim P'C' \sim BB'$. By transitivity of $\sim$, we infer $CP' \sim CP$. So, by A2, $P = P'$. Therefore, $BB' \sim PC$. Now, we calculate: $BC \sim AB = AB' + B'B \sim$ (by A3) $B'C' + C'P \sim B'P$ ☐

Remark 1   Assuming only Axioms A1 to A3 and Corollary A'4, without any axiom on the multiplication, we can prove that the elementary scale invariance axiom is equivalent to the conjunction of the unique translation of arrows law and parallelogram law.

Fig. 3 Illustration for the parallelogram law
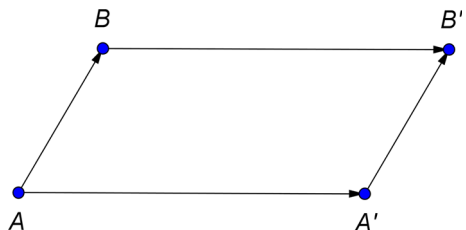
Remark 2 Until now we haven't used Axiom A1.1 (reflexivity). Indeed we can prove it: by the unique translation of arrows law T3, for given points A and B there is the unique C such that $AB \sim AC$. From A2 it follows $B = C$. Therefore, $AB \sim AB$.

Remark 3 The parallelogram law is very efficient algebraically in proofs. For example, assuming only the parallelogram law, we can prove the equivalence of: (i) A1.1 (reflexivity of $\sim$) and the next corrolary C5 (all arrows of the type AA are equivalent), (ii) A1.2 (symmetry of $\sim$) and A3.1 (invariance under $\sim$ of inverting arrows) , (iii) A1.3 (transitivity of $\sim$) and A3.2 (invariance under $\sim$ of adding arrows).

Corollary C5 $\quad \forall A, B \quad AA \sim BB$.

Proof From reflexivity of $\sim$, we infer $AB \sim AB$. By the parallelogram law, it follows that $AA \sim BB$. □

4 Elimination of the Multiplication of an Arrow by a Number

Here, I will sketch how can we define the multiplication of an arrow by a number and establish basic properties of the operation. I continue exposition from the second section, based on Axioms A1 to A3, neglecting the third section. In the second section we have described the relation $\sim$ between arrows originating from the same point. How can we compare arrows originating from different points? There is an obvious case: we can translate an arrow AB along itself. In that way, by the homogeneity principle, we can reproduce the position of B relative to A in the point B. This is the content of the Axiom A'4 (which we have proved in the second section from the multiplication axioms).

Axiom A$'$4 $\quad \forall A, B \; \exists C \quad AB \sim BC$.

An immediate consequence is that we can translate AB in the opposite direction:

Corollary C1 $\quad \forall A, B \; \exists D \quad DA \sim AB$

We have proved this in the previous section using only A'4, A3.2 and A1.2.

As the next axiom we will use the elementary scale invariance axiom (which we have also proved in the previous section from the multiplication axioms)

Axiom A$'$5 (*the elementary scale invariance axiom*) $\quad AB \sim BC$ and $AB' \sim B'C' \; \rightarrow \; \exists \, P \quad CP \sim PC' \sim BB'$.

These two axioms and Axioms A1 to A3 yield the unique translation of arrows law T3 and the parallelogram law T4. I have used nothing else in the proofs of these laws in the previous section.

Because the unique translation of arrows law enables us to add arbitrary arrows, we can add an arrow to itself, as a special case. It enables us to multiply the arrow by a positive natural number:

$$1 \cdot AB = AB, \ 2 \cdot AB = AB + AB, \ 3 \cdot AB = 2 \cdot AB + AB, \ldots$$

In that way we can define recursively the multiplication of an arrow by an arbitrary positive natural number. Then, we can define the multiplication of an arrow by an arbitrary integer:

$$0 \cdot AB = AA, \ (-n) \cdot AB = n \cdot (-AB)$$

where n is a positive natural number.

Furthermore, we can introduce axiomatically (and justify it by the symmetry principles and an idea of divisibility of space) the existence of the midpoint P of an arrow AB, as the point for which $AP \sim PB$,

Axiom A′6 $\quad \forall A, B \ \exists! P \ \ AP \sim PB$.

and then define multiplication of the arrow by $\frac{1}{2}$:

$$\frac{1}{2} \cdot AB = AP$$

By repeating bisection we can define recursively multiplication of an arrow by $\frac{1}{2^n}$. On that basis we can define multiplication of an arrow by an arbitrary rational number of the form $\frac{m}{2^n}$, where m is an integer and n is a positive natural number.

We can extend the multiplication to all real numbers using an appropriate axiom of continuity. For example,

Axiom A′7 *Let $p_2(A, B)$, for $A \neq B$, be the set of all points C such that $AC = \lambda \cdot AB$, where λ is rational number of the form $\frac{m}{2^n}$. Let's define the order relation $<_2$ on $p_2(A, B)$: for $AC = \lambda_C \cdot AB$ and $AD = \lambda_D \cdot AB, C < D \ \leftrightarrow \ \lambda_C < \lambda_D$. Then there is the unique linearly ordered extension $p(A, B)$ of $p_2(A, B)$ such that every bounded above non empty subset of $p(A, B)$ has supremum.*

Now, we can define multiplication of an arrow by an arbitrary real number λ. Because for every $\lambda > 0$ there exists the unique strictly increasing sequence $\frac{m_n}{2^n}$ such that $\lambda = \lim_{n \to \infty} \frac{m_n}{2^n}$, we define

$$\lambda \cdot AB = A \sup\{C_n | AC_n = \frac{m_n}{2^n} \cdot AB\}$$

$$(-\lambda) \cdot AB = -(\lambda) \cdot AB$$

By the homogeneity principle, all these operations are invariant under $\sim$:

Axiom A′8 *The operations of bisection and supremum are invariant under $\sim$.*

From these axioms all the axioms for the multiplication can be proved.

5 Vectors

Axiom A1 enables us to define vectors. Since, by A1, $\sim$ is an equivalence relation, it classifies arrows (pairs of points) into classes of mutually equivalent arrows. We

define **vectors** as these equivalence classes. The set of all vectors will be denoted $\vec{S}$. To every pair of points AB we will associate the vector $\overrightarrow{AB}$, the equivalence class to which AB belongs:

$$\overrightarrow{AB} = \{CD|CD \sim AB\}$$

So, $\rightarrow$ maps pairs of points to vectors: $\rightarrow: S^2 \rightarrow \vec{S}$.

For every pair of points (arrows) from a vector (an equivalence class) we say that it **represents** the vector. Thus, for example, a pair AB represents the vector $\overrightarrow{AB}$.

Theorem W1 *For every point A the function $X \mapsto \overrightarrow{AX}$ is a bijection from set of points S onto set of vectors $\vec{S}$.*

Proof The claim is an immediate consequence of the unique translation of arrows law (T3) $\square$.

For a given point A the inverse function of bijection $X \mapsto \overrightarrow{AX} = \vec{x}$ maps every vector $\vec{x}$ to the point X which we will denote $X = A + \vec{x}$. Thus,

$$A + \vec{x} = X \ \leftrightarrow \ \overrightarrow{AX} = \vec{x}$$

We will transfer operations with arrows into operations with corresponding vectors, in a way invariant under relation $\sim$.

null vector $\ \ \vec{0} = \overrightarrow{AA}$

By Corollary C5, the definition is correct because it does not depend on the choice of a point A.

inverse vector $\ \ -\overrightarrow{AB} = \overrightarrow{BA}$

By Axiom A3.1 (invariance under $\sim$ of inverting arrows), the definition is correct because it does not depend on the choice of an arrow AB.

Addition of vectors (W2) $\ \ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

By Axiom A3.2 (invariance under $\sim$ of addition of arrows) the definition is correct because it does not depend on the choice of arrows which represent vectors.

Vectors will be denoted by letters with "arrows", for example, $\vec{a}$.

Theorem W3 *Addition of vectors makes set of all vectors into a commutative group*:

1. $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$
2. $\vec{a} + \vec{0} = \vec{a}$
3. $\vec{a} + (-\vec{a}) = \vec{0}$
4. $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Proof We will represent vectors by the corresponding arrows. Let $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{BC}$ and $\vec{c} = \overrightarrow{CD}$. By the definitions of addition of vectors, null vector and inverse vector, the claims we want to prove follow from the corresponding claims about arrows. □

1. $(AB + BC) + CD = AC + CD = CD, AB + (BC + CD) = AB + BD = CD$
2. $AB + BB = AB$
3. $AB + (-AB) = AB + BA = AA$
4. Commutativity of vector addition is a consequence of the parallelogram law (T4). Let's choose a point D such that $AD \sim BC$. By the parallelogram law, then $AB \sim DC$ (Fig. 4).

Now, we will calculate the left and right side of the identity we want to prove by "going" along the sides of the parallelogram from the vertex A to the opposite vertex C in two different ways, via the point B and via the point D:

$$\vec{a} + \vec{b} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$
$$\vec{b} + \vec{a} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$$

Hence, $\vec{a} + \vec{b} = \vec{b} + \vec{a}$. □

Since the multiplication of an arrow by a number is invariant under relation $\sim$ (Axiom A5), we can transfer the operation into the **multiplication of a vector by a number**:

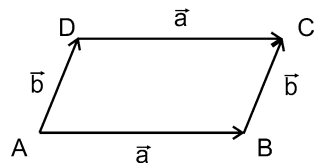$$\lambda \cdot \overrightarrow{AB} = \overrightarrow{\lambda \cdot AB}$$

Because of the invariance, we can transfer all the properties of the multiplication of an arrow by a number into the properties of the multiplication of a vector by a number:

Theorem W4 *Multiplication of a vector by a number has the following properties*:

1. $1 \cdot \vec{a} = \vec{a}$
2. $(\lambda + \mu) \cdot \vec{a} = \lambda \cdot \vec{a} + \mu \cdot \vec{a}$
3. $\lambda \cdot (\mu \cdot \vec{a}) = (\lambda \cdot \mu) \cdot \vec{a}$
4. $\lambda \cdot (\vec{a} + \vec{b}) = \lambda \cdot \vec{a} + \lambda \cdot \vec{b}$

Proof Let $\vec{a} = AB$ and $\vec{b} = BC$

Fig. 4 Illustration for the proof of the commutativity of vector addition

1. $1 \cdot \overrightarrow{a} = \overline{1 \cdot AB} = \overrightarrow{AB} = \overrightarrow{a}$

2. $(\lambda + \mu) \cdot \overrightarrow{a} = (\lambda + \mu) \cdot \overrightarrow{AB} = \overline{(\lambda + \mu) \cdot AB} = \overline{\lambda \cdot AB + \mu \cdot AB}$
    $= \overline{\lambda \cdot AB} + \overline{\mu \cdot AB} = \lambda \cdot \overrightarrow{AB} + \mu \cdot \overrightarrow{AB} = \lambda \cdot \overrightarrow{a} + \mu \cdot \overrightarrow{a}$

3. $\lambda \cdot (\mu \cdot \overrightarrow{a}) = \lambda \cdot (\mu \cdot \overrightarrow{AB}) = \lambda \cdot \overline{\mu \cdot AB} = \overline{\lambda \cdot (\mu \cdot AB)} = \overline{(\lambda \cdot \mu) \cdot AB} = (\lambda \cdot \mu) \cdot$
    $\overrightarrow{AB} = (\lambda \cdot \mu) \cdot \overrightarrow{a}$

4. $\lambda \cdot (\overrightarrow{a} + \overrightarrow{b}) = \lambda \cdot (\overrightarrow{AB} + \overrightarrow{BC}) = \lambda \cdot \overline{AB + BC} = \overline{\lambda \cdot (AB + BC)}$
    $= \overline{\lambda \cdot AB + \lambda \cdot BC} = \overline{\lambda \cdot AB} + \overline{\lambda \cdot BC} = \lambda \cdot \overrightarrow{AB} + \lambda \cdot \overrightarrow{BC} = \lambda \cdot \overrightarrow{a} + \lambda \cdot \overrightarrow{b}$

$\square$

All previously established propositions with label W say that the space of vectors $\overrightarrow{S}$ together with operations of addition of vectors and multiplication of a vector by a number has the structure of vector space, and that the space of points S together with the space of vectors $\overrightarrow{S}$ and the mapping $AB \mapsto \overrightarrow{AB}$, that is to say, the structure

$$(S, \overrightarrow{S}, \rightarrow, +, \cdot), \quad \text{where} \quad \rightarrow : S^2 \to \overrightarrow{S}, \ + : \overrightarrow{S}^2 \to \overrightarrow{S}, \ \cdot : \mathbb{R} \times \overrightarrow{S} \to \overrightarrow{S},$$

is Weyl's structure of affine space. W propositions are precisely Weyl's axioms for the affine part of Euclidean geometry. Conversely, starting from Weyl's structure of affine space, for which W propositions are valid, we could define in a standard way the equivalence of arrows, addition of arrows and multiplication of an arrow by a number, and prove that for such a defined structure

$$(S, \sim, +, \cdot), \quad \text{where} \ S \neq \emptyset, \ \sim \ \subseteq S^2 \times S^2, \ + : S^2 \times S^2 \to S^2, \ \cdot : \mathbb{R} \times S^2 \to S^2$$

all A propositions are valid. Hence,

A axioms are equivalent to W axioms in this affine layer of Euclidean geometry.

Let's note that in the structure $(S, \sim, +, \cdot)$ addition $+$ is defined and multiplication $\cdot$ can be defined if we choose to introduce multiplication of a vector gradually, first by natural numbers, then by integers and rational numbers, and finally by real numbers. It means that there is essentially only one primitive term, the relation of equivalence $\sim$ between ordered pairs of points.

In the next section we will use a few propositions of affine geometry concerning parallelograms and projections. It is well known that the propositions follow from W (and so from A) axioms and they will not be proved here.

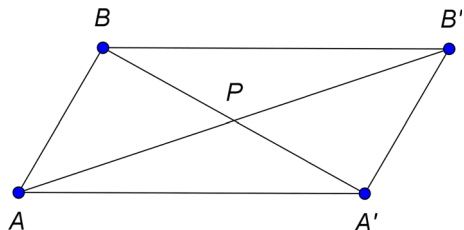**Fig. 5** Illustration for characterization of a parallelogram by diagonals

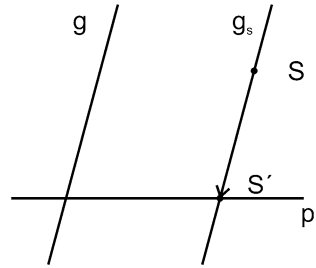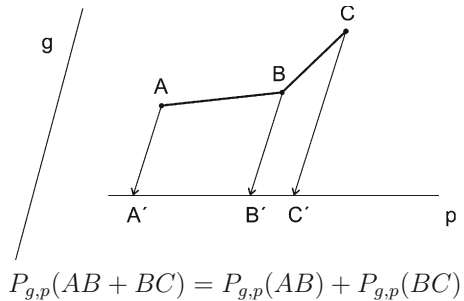**Fig. 6** Illustration for the definition of a projection

Fig. 7 The sum of arrows is invariant under projection

$$P_{g,p}(AB + BC) = P_{g,p}(AB) + P_{g,p}(BC)$$

We will define a parallelogram in the usual way, as a quadrilateral *ABCD* such that $AB \sim DC$. We will need the following result about parallelograms:

characterization of a parallelogram by diagonals *A quadrilateral is a parallelogram if and only its diagonals bisect each other* (Fig. 5).

$$AB \sim A'B' \ \leftrightarrow \ AP \sim PB' \ \wedge \ BP \sim PA'$$

The definition of a projection is based on the following proposition. *In a fixed plane, for every (straight) line g and point S there is a unique line g_S through S which does not intersect g or is equal to g* (*when S lies on g*). *The line g_S intersects every line p which intersects g in a unique point S′. The mapping $S \mapsto S′$ maps every point of the plane onto a point of the line p. This mapping is termed* **projection $P_{g,p}$** *onto the line p generated by the line g* (Fig. 6).

We will need two **properties of projection**:

1. *Projection maps the sum of arrows onto the sum of the projections of the arrows* (Fig. 7):
2. *Projection maps a λ times longer arrow into a λ times longer projection of the arrow* (Fig. 8):

Fig. 8 The multiplication of an arrow by a number is invariant under projection

$$P_{g,p}(\lambda AB) = \lambda P_{g,p}(AB)$$
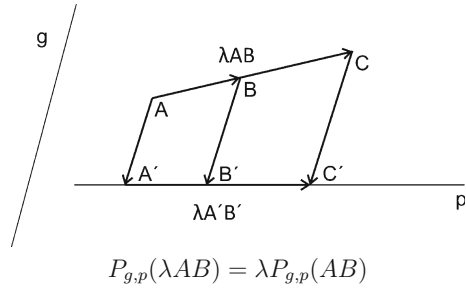
6 Length

A basic geometric measure is the measure of distance between points A and B, the function $|\ \ | : S^2 \to \mathbb{R}$. This is the next and the final primitive term of the theory I develop here. A real number $|AB|$ will be termed the **length** of the arrow AB or **distance** from the point A to the point B.

By the homogeneity of space the length of an arrow must be invariant under equivalence relation $\sim$:

Axiom A8 $\quad AB \sim CD \ \to \ |AB| = |CD|$.

By the very idea of measuring distance:

Axiom A9.1 $\quad |AA| = 0$.

Every point $B \neq A$ determines a direction in which we can go from A. Because of the isotropy of space, the algebraic sign of distance must be always the same – distance must be always negative or always positive or always zero. The zero case gives a trivial measure which does not make any difference between arrows, so, it is a useless measure. Thus, the two other possibilities remain. Technically speaking they are mutually equivalent choices, but by the very idea of measuring it is natural to choose a positive algebraic sign:

Axiom A9.2 $\quad B \neq A \ \to \ |AB| > 0$. (*positive definiteness*)

By the isotropy of space we also have:

Axiom A9.3 $\quad |AB| = |BA|$.

For every direction from a point A determined with a point $B \neq A$ we already have a measure of distance. If we take AB as a unit of measure, than we can take the number $\lambda > 0$ as a measure of distance of AC where $AC = \lambda AB$. Note that such a choice of measure along every direction need not be isotropic. However, along every direction the measure of distance $A, B \mapsto |AB|$ must be in accordance with this λ measuring (although it must be more than this):

Axiom 10 $\quad |\lambda AB| = \lambda |AB|$, for $\lambda > 0$,

We can express Axioms A9.1, A9.3 and A10 in a uniform way by the next equivalent proposition:

Fig. 9 Two points *A* and *B* on the circle and the unique line p(*A*, *B*) through these points
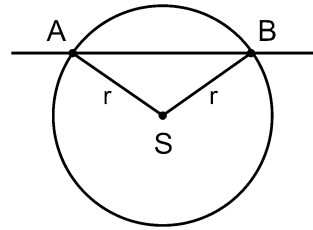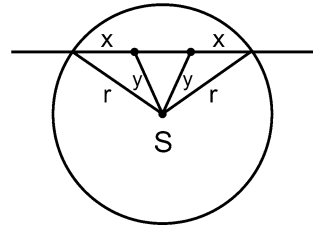

Fig. 10 Symmetry of the distance function

Theorem 7 (*compatibility of distance with multiplication*) $|\lambda AB| = |\lambda||AB|$, *for every real number* λ.

Proof By A10, the claim is valid for $\lambda > 0$. By A9.1 the claim is valid for $\lambda = 0$. By A9.3 the claim is valid for $\lambda = -1$. Thus, we must prove the claim for the remaining negative values of λ. Let $\lambda < 0$. Then $\lambda = -\mu$, where $\mu > 0$. Now we calculate using what we have already proven:

$$|\lambda AB| = |(-1)\mu AB| = |-1||\mu AB| = |-1||\mu||AB| = |(-1)\mu||AB| = |\lambda||AB| \quad \square$$

The description of distance function we have achieved until now enables us to compare distances in a given direction with distances in the opposite direction and with distances in parallel directions. What remains is to solve the main problem: how to compare distances along arbitrary directions in an isotropic way. Let's take, in a given plane, along every direction from a point *S*, a point at a fixed distance $r > 0$ from *S*. The set of such points is the **circle** with center *S* and radius r, $C(S, r) = \{T : |ST| = r\}$. Let's choose two points *A* and *B* on the circle and consider the unique line p(*A*, *B*) through these points (Fig. 9):

Let's take an arbitrary point *T* on the line p(*A*, *B*) and consider how the distance $d(T)$ from *T* to the center *S* of the circle varies with the choice of *T*. Thereby, we will use the idea of continuity of space and of continuity of function $d(T)$. Because of the isotropy of space, the function $d(T)$ must be symmetric in the position of *T* relative to the points *A* and *B* (directions *SA* and *SB*). For example, values of the function in the points *A* and *B* are the same (equal to *r*). Also, the function must have the same value in a point we reach when we move a certain distance from *A* to *B* as well as in a point we reach when we move the same distance from *B* to *A*: $d(A + \lambda \overrightarrow{AB}) = d(B + \lambda \overrightarrow{BA})$ (Fig. 10):

Because of this symmetry, the function $d(T)$ must have a local extreme value in the midpoint of *AB*. To determine more precisely the character of the extreme point

Fig. 11 Illustration for the argument that the midpoint P is a unique point of the global minimum of the distance function on the line p(A, B)
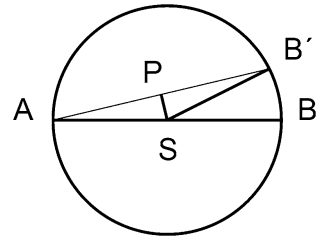

Fig. 12 The midpoint P of AB is the point on the line nearest to the center of the circle
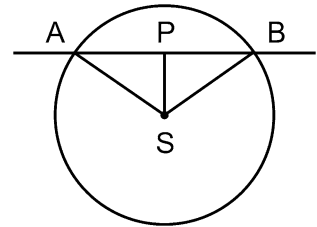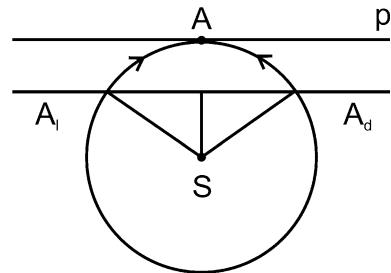

Fig. 13 Illustration for the argument that the point A must be the point on p nearest to the center of the circle

we will exploit knowledge of a special case, when the points A and B are diametrically opposite on the circle, that is to say, when the center S of the circle lies on $p(A, B)$. In that case, if we "move" a point T from A to B (or from B to A), the distance $d(T)$ from the center S of the circle decreases and it is smallest in the midpoint (S). Furthermore, if we move T from A in the direction opposite to the direction to B (or from B in the direction opposite to the direction to A), the distance increases. Therefore, the midpoint S is a unique point of the global minimum of the function $d(T)$. If we drag the point B slightly along the circle into the point B', the center S of the circle will no longer be on the line $p(AB')$, but, because of continuity, the behaviour of the function $d(T)$ will remain the same. That is to say, the midpoint P of AB' will remain a unique global minimum of the function on the line (Fig. 11):

Because of continuity, for every two points A and B' on the circle the function $d(T)$ will have a unique global minimum on line $p(AB')$ exactly in the midpoint of AB':

Axiom 11 *If a line has two common points with a circle, points A and B, then the midpoint P of AB is the point on the line nearest to the center of the circle* (Fig. 12).

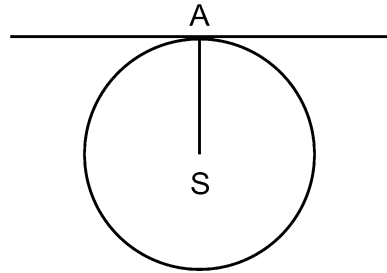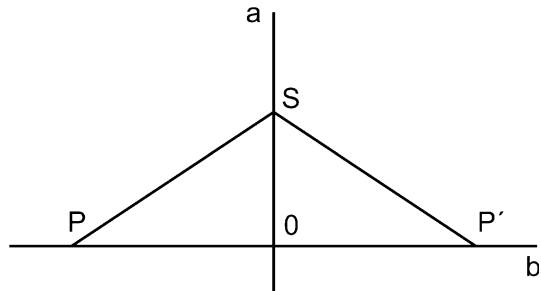**Fig. 14** The point *A* is the point on the line nearest to the center of the circle

Fig. 15 The point *O* is the midpoint of *PP′*

From the axiom it follows immediately that a line can not have more than two common points with a circle. If, besides *A* and *B*, there was a third common point *X*, by A11, the midpoint of *AB* and the midpoint of *AX* would be points on a line nearest to the center of the circle. Then, by uniqueness of the nearest point, *AB* and *AX* would have the same midpoint. Thus, we would get a contradiction, that *X* = *B*.

Let a line *p* have exactly one common point with a circle, a point *A*. If we drag the point *A* slightly along the circle in one direction onto a point *Al*, and in another direction onto a point *Ad*, then the line *p* is dragged onto the line *p*(*Al*, *Ad*). By Axiom 11 the midpoint *P* of *AB* is the point on *p*(*Al*, *Ad*) nearest to the center of the circle. By continuity of space, the point *A* must be the point on *p* nearest to the center of the circle (Fig. 13):

Axiom A12 *If a line has exactly one common point with a circle, then the common point is the point on the line nearest to the center of the circle* (Fig. 14).

Theorem 7 *For every point S not on a line p there is a unique point P on p which is the point on p nearest to S.*

Proof Let *A* be a point on a line *p* and let its distance to *S* be *r*. Thus, *A* is a common point of the line *p* and the circle with center *S* and radius *r*. If the line *p* does not have another common point with the circle, then, by A12, *A* is a point on *p* nearest to *S*. If line *p* has another common point *B* with the circle, then, by A11, the

Fig. 16 The point O is the midpoint of SS'
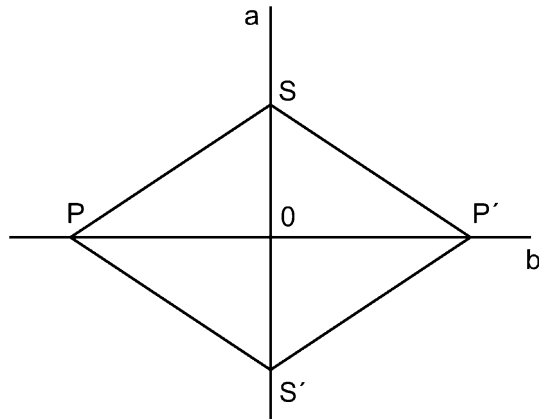
midpoint P of AB is the point on p nearest to S. Thus, we have proved the existence of the nearest point. The nearest point is unique by its very definition. □

The direction from a point not on a line to its nearest point on the line is a perpendicular direction to the line. Formally, we define that a line a is **perpendicular** to a line b, in symbols $a \perp b$, if a intersects b and there is a point S on a which is not on b such that the intersection of a and b is the point on b nearest to S.

Theorem 8 *For lines a and b, if $a \perp b$ then $b \perp a$.*

Proof By definition of perpendicularity, there is a point S on a such that the intersection of a and b, the point O, is the point on b nearest to S. Let P be a point on b different from O. Since its distance to S is greater than the distance of O to S, there is another point P' on the line b equally distant as P to S (Fig. 15). On the contrary, by A12, P would be the point on b nearest to S, hence, it would be $P = O$, which is in a contradiction of the choice of P. Since O is the point on b nearest to S, it is, by A11, the midpoint of PP'.

By the uniqueness of the translation of arrows law T3. there is a point S' on the line a such that $SO \sim OS'$ (Fig. 16). Thus, the point O is the midpoint of SS', too. Therefore, diagonals of the quadrilateral $SPS'P'$ bisect each other.

By the theorem of affine geometry on the characterization of a parallelogram by diagonals, the quadrilateral is a parallelogram. Since it is a parallelogram, the opposite sides have the same length. However, by the choice of the point P', the neighbouring sides have the same length, too. Therefore, all sides of the quadrilateral have the same length. Thus, PS and PS' have the same length. Hence, by A11, the point O is the point on a nearest to P. By the definition of perpendicularity, it means that the line b is perpendicular to the line a.

Theorem 9 *A line b is perpendicular to a line a if and only if they have a common point O and for every point P on line b the intersection O is the point on the line a nearest to P.*
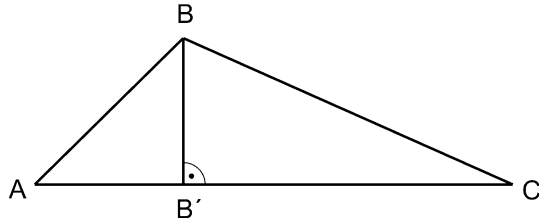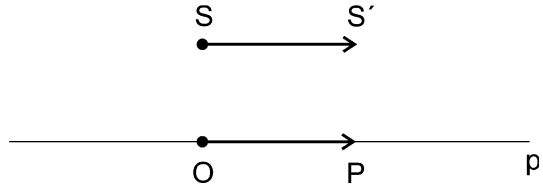
Fig. 17 $|AB| + |BC| \geq |AC|$

Fig. 18 $SS' \sim OP$

Proof One direction is trivial: if for every point P on line b the intersection O is the point on line a nearest to P then that is true for a particular P on b, so, by definition of perpendicularity, $b \perp a$. The opposite direction is the immediate consequence of the previous theorem and its proof. Namely, by the theorem, from $b \perp a$ follows $a \perp b$. In the proof of the previous theorem, from that assumption it is proved that for every point P on the line b the intersection O is the point on the line a nearest to P. □

Corollary 10 (*the triangle inequality theorem*): $|AB| + |BC| \geq |AC|$. *The equality is valid if and only if the point B is on AC.*

Proof We will prove only the most important case (Fig. 17), when the point B is not on line $p(A, C)$ and the line through B perpendicular to $p(A, C)$ intersects $p(A, C)$ at a point B' which is on AC (by Theorem 7, the perpendicular line exists).

By Theorem 8, the line $p(A, C)$ is perpendicular to the line $p(B, B')$. By Theorem 9, B' is the point on $p(B, B')$ nearest to points A and C. Hence, $|AB| > |AB'|$ and $|BC| > |B'C|$. Thus, $|AB| + |BC| > |AB'| + |B'C| = |AC|$. □

Theorem 9 enables us to show that for every point and every line there is a unique line through the point perpendicular to the given line .

Theorem 11 *For a point S not on a line p there is a unique line through the point S perpendicular to the line p.*

Proof By Theorem 7, there is a point O on the line p nearest to S. By the definition of perpendicularity, the line $a = p(S, O)$ is perpendicular to p. Thus, we have proved the existence of the perpendicular line. Let S lie on another perpendicular line b. By Theorem 10, the intersection O' of the line b and the line p is the point on p nearest to S. By the uniqueness of the nearest point, $O' = O$. Thus, b as well as a contains S and O, hence $b = a$. □

Fig. 19 Illustration for the
proof of the uniqueness of the
perpendicular line
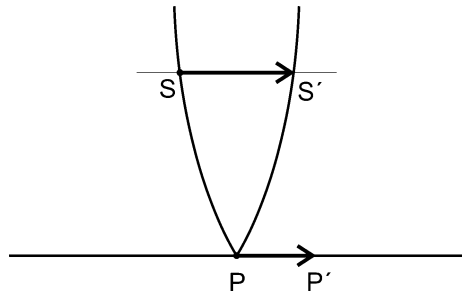
Theorem 12 *For every point P on a line p there is a unique line through
P perpendicular to p.*

Proof To show the existence of the perpendicular line let's choose a point S not on
p. If the intersection O of the line through S perpendicular to p with line p is just the
point P then we have found the perpendicular line. If it is not so, let S' be a point
such that $SS' \sim OP$ (Fig. 18):

Because of the homogeneity invariance of length (Axiom A8), since O is a point
on p nearest to S, the point P is the point on p nearest to S'. Hence, P is on the line
through S' perpendicular to line p.

To prove the uniqueness of the perpendicular line, let's suppose the contrary, that
there are two lines through the point P perpendicular to the line p. Let S be a point
on one of perpendicular lines, the line a, but not on p and not on the other
perpendicular line, the line b. By affine geometry, there is a unique line through S
parallel to p. Let S' be the intersection point of the parallel line with another
perpendicular line b (Fig. 19):

Let P' be a point on line p such that $PP' \sim SS'$ Let's observe that $P' \neq P$. Because
of the homogeneity invariance of length, since P is the point on p nearest to point S,
so the point P' is a point on p nearest to point S'. However, since P is on the line
through S' perpendicular to p, P is the nearest point. Thus, we get a contradiction,
that $P' = P$. Therefore, there is no more than one line through the point P
perpendicular to the line p. □

We will define the **orthogonal projection** of a point S to a line p to be the
intersection of the line p and the unique line through S perpendicular to p, that is to
say, the point on line p nearest to point S. The next theorem says that it is indeed a
projection.

Theorem 13 *All lines perpendicular to a line p in a given plane containing p are
mutually parallel.*

Proof In a given plane, lines a and b are parallel if they have no common points,
or they are the same lines. Let a and b be lines perpendicular to a line p in a given
plane containing p. If a and b have no common points they are parallel. Let them
have a common point S. By Theorems 11 (if S is not on p) and 12 (if S is on p) there
is a unique line through S perpendicular to p. This means that $a = b$. So, in this case,
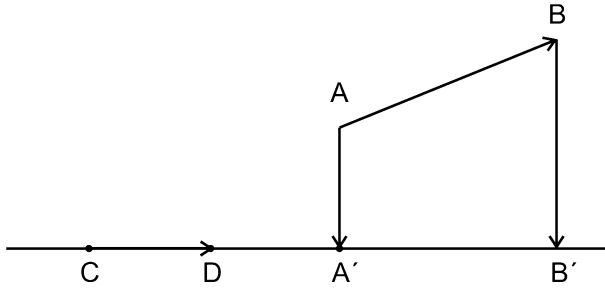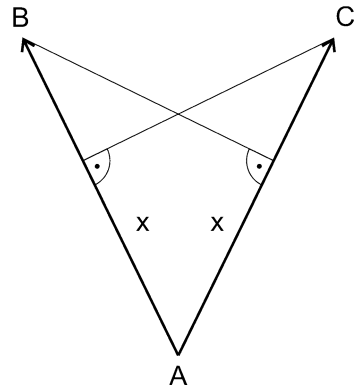a and b are parallel, too. □

Fig. 20 Orthogonal projection of an arrow

Fig. 21 Equally long arrows *AB* and *AC* have equally long mutual orthogonal projections

Therefore, in a given plane, parallel lines generated by a line *g* perpendicular to a line *p* are precisely lines perpendicular to the line *p*.

Orthogonal projection enables us to define the scalar orthogonal projection of an arrow onto another arrow. Let $C \neq D$, and let points *A* and *B* be orthogonally projected on line $p(CD)$ into points A' and B' (Fig. 20). Then $A'B' \sim \alpha CD$ for some real number α.

We define the **scalar orthogonal projection** of the arrow *AB* onto the arrow *CD* to be the number $\alpha|CD|$. In simpler terms, it is just the $\pm$ length of the orthogonal projection of the arrow *AB* onto the line $p(CD)$, where the sign is $+$ if the projection is in the direction of CD, $-$ otherwise. In the extreme case of null arrow *CC* it is convenient to take zero for the value of the scalar projection on *CC*. We will denote AB_{CD} as the scalar projection of *AB* onto *CD*.

For two equally long arrows with the same initial point, because of the isotropy of space, the scalar projection of the first arrow on the second arrow must be the same as the scalar projection of the second arrow on the first arrow. This is the content of the final axiom:

Axiom A13 $|AB| = |AC| \rightarrow AB_{AC} = AC_{AB}$ (Fig. 21).

Now, we can define the **scalar product of arrows** AB and CD. It is the product of the scalar projection of the arrow AB onto CD and the length of the arrow CD. More formally:

$$AB \cdot CD = AB_{CD} \cdot |CD|$$

This operation is invariant under $\sim$ relation:

Theorem 14 $A'B' \sim AB \ \wedge \ C'D' \sim CD \ \rightarrow \ A'B' \cdot C'D' = AB \cdot CD$.

Proof It is the consequence of the properties of projection that (i) equivalent arrows are projected onto equivalent arrows and (ii) projections generated by the same line onto parallel lines map an arrow onto two mutually equivalent arrows. $\square$

Theorem 15 (*properties of the scalar product of arrows*)

1. $A \neq B \ \rightarrow AB \cdot AB > 0$
2. $(AB + A'B') \cdot CD = AB \cdot CD + A'B' \cdot CD$
3. $(\lambda AB) \cdot CD = \lambda(AB \cdot CD) \quad AB \cdot (\lambda CD) = \lambda(AB \cdot CD)$
4. $AB \cdot CD = CD \cdot AB$

Proof

1. $AB \cdot AB = AB_{AB} \cdot |AB| = |AB| \cdot |AB| > 0$, because, by A9.2, for $A \neq B |AB| > 0$.
2. For the proof it is crucial that, by Theorem 13, orthogonal projection is a special type of projection. From affine geometry it is known that for every projection it is valid that $(AB + A'B')_{CD} = AB_{CD} + A'B'_{CD}$. So, $(AB + A'B') \cdot CD = (AB + A'B')_{CD} \cdot |CD| = (AB_{CD} + A'B'_{CD}) \cdot |CD| = AB_{CD} \cdot |CD| + A'B'_{CD} \cdot |CD| = AB \cdot CD + A'B' \cdot CD$
3. Because orthogonal projection is a special type of projection and for every projection it is valid that $(\lambda AB)_{CD} = \lambda(AB_{CD})$, we have $(\lambda AB) \cdot CD = (\lambda AB)_{CD} \cdot |CD| = \lambda(AB_{CD}) \cdot |CD| = \lambda(AB \cdot CD)$ Thus, the first claim is proved. The second claim is an immediate consequence of definition of the scalar product of arrows: $AB \cdot (\lambda CD) = AB_{\lambda CD} \cdot |\lambda CD| = |\lambda| AB_{\lambda CD} \cdot |CD| = \lambda AB_{CD} \cdot |CD| = \lambda(AB \cdot CD)$
4. We will express the arrow AB as a stretched arrow of the arrow $C'D'$ which has length equal to the length of CD, $AB = \lambda C'D'$. Now, we will apply the previously proved proposition, Axiom A13, and invariance under $\sim$ of the scalar product of arrows (Theorem 14):

$$\begin{aligned} AB \cdot CD &= (\lambda C'D') \cdot CD = \lambda(C'D' \cdot CD) = \lambda(CD \cdot C'D') \\ &= CD \cdot (\lambda C'D') = CD \cdot AB \end{aligned}$$

$\square$

The **scalar product of vectors** is defined by arrows which represent vectors:

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = AB \cdot CD$$

Since scalar product of arrows is invariant under $\sim$ (Theorem 14) the definition is correct, that is to say, it doesn't depend on the choice of arrows representing vectors.

Theorem W5 (*properties of the scalar product of vectors*)

1. $\overrightarrow{a} \neq \overrightarrow{0} \;\rightarrow\; \overrightarrow{a} \cdot \overrightarrow{a} > 0$
2. $(\overrightarrow{a} + \overrightarrow{b}) \cdot \overrightarrow{c} = \overrightarrow{a} \cdot \overrightarrow{c} + \overrightarrow{b} \cdot \overrightarrow{c}$
3. $(\lambda \overrightarrow{a}) \cdot \overrightarrow{b} = \lambda (\overrightarrow{a} \cdot \overrightarrow{b})$
4. $\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a}$

Proof These claims, by definition of the scalar product of vectors, are reduced to corresponding claims about the scalar product of arrows, which were proved in the previous theorem. □

Proposition W5 about the scalar product of vectors, together with W propositions about the affine structure of space, form Weyl's axiomatics of Euclidean geometry. Conversely, by Weyl's axioms we can define in a standard way the length of an arrow and deduce all A axioms about length. This means that A axioms of the structure of set of points S with relation $\sim$ between pairs of points, addition of pairs, multiplication of a pair by a number and distance function of a pair:

$$(S, \sim, +, \cdot, |\;|), \quad \text{where } S \neq \emptyset, \ \sim \ \subseteq S^2 \times S^2, \ + : S^2 \times S^2 \rightarrow S^2, \ \cdot :$$
$$\mathbb{R} \times S^2 \rightarrow S^2, \ |\;| : S^2 \rightarrow \mathbb{R}$$

are equivalent to Weyl's axioms of the corresponding structure of the set of points and the set of vectors together with the operation from pairs of points to vectors, addition of vectors, multiplication of a vector by a number and the scalar product of vectors:

$$(S, \overrightarrow{S}, \longrightarrow, +, \cdot, \cdot), \quad \text{where } S \neq \emptyset, \ \longrightarrow : S^2 \rightarrow \overrightarrow{S}, \ +, \cdot : \overrightarrow{S}^2 \rightarrow \overrightarrow{S}, \ \cdot :$$
$$\mathbb{R} \times \overrightarrow{S} \rightarrow \overrightarrow{S}$$

Appendix

The primitive terms of the system of axioms are (i) equivalence of pairs of points (arrows): $AB \sim CD$, with the intuitive meaning that the position of the point B relative to the point A is the same as the position of the point D relative to the point C, (ii) multiplication of a pair of points (an arrow) by a real number: $\lambda, A, B \mapsto \lambda \cdot AB$, with the intuitive meaning of stretching the arrow and of iterative addition of the same arrow, and (iii) distance between points: $A, B \mapsto |AB| \in \mathbb{R}$.

Axiom (A1) $\sim$ *is an equivalence relation.* (by the very idea *to be in the same relative position*)

Axiom (A2) $AB \sim AC \rightarrow B = C$. (by the very idea of the relative position of points)

The definitions od inverse arrow and of adding arrows:

inverting arrow $AB \mapsto -AB = BA$

inverting arrow $AB, BC \mapsto AB + BC = AC$

Because of Axiom A2 we can extend addition of arrows:

generalized addition of arrows $AB + CD = AB + BX$, where $BX \sim CD$, under the condition that there is such a point X.

Axiom (A3.1) $AB \sim A'B' \rightarrow BA \sim B'A'$. (by the homogeneity principle)
Axiom (A3.2) $AB \sim A'B' \wedge BC \sim B'C' \rightarrow AC \sim A'C'$. (by the homogeneity principle)
Next axioms describe multiplication of an arrow by a real number.
Axiom (A4) $\forall \lambda, A, B \exists C \ \lambda \cdot AB = AC$.
(by the very idea of the multiplication as stretching arrows)
Axiom (A5) $AB \sim CD \rightarrow \lambda AB \sim \lambda CD$.
(by the homogeneity principle)
Axiom (A6.1) $1 \cdot AB = AB$.
(by the very idea of the multiplication as addition of the same arrow)
Axiom (A6.2) $\lambda \cdot AB + \mu \cdot AB = (\lambda + \mu) \cdot AB$.
(by the homogeneity principle and by the very idea of the multiplication as iterative addition of the same arrow)
Axiom (A6.3) $\lambda \cdot (\mu \cdot AB) = (\lambda \cdot \mu) \cdot AB$.
(by the very idea of the multiplication as iterative addition of the same arrow)
Axiom (A7) (*the scale invariance axiom*)
If $AC = \lambda \cdot AB$ and $AC' = \lambda \cdot AB'$ then $CC' \sim \lambda \cdot BB'$.
(by the scale invariance principle)
Next axioms describe the distance function.
Axiom (A8) $AB \sim CD \rightarrow |AB| = |CD|$.
(by the homogeneity principle)
Axiom (A9.1) $|AA| = 0$.
(by the very idea of measuring distance)
Axiom (A9.2) $B \neq A \rightarrow |AB| > 0$. (*positive definiteness*)
(by the isotropy principle)
Axiom (A9.3) $|AB| = |BA|$.
(by the homogeneity principle)
Axiom (A10) $|\lambda AB| = \lambda |AB|$, *for* $\lambda > 0$.
(by the very idea of measuring along AB)

The definition of a circle:
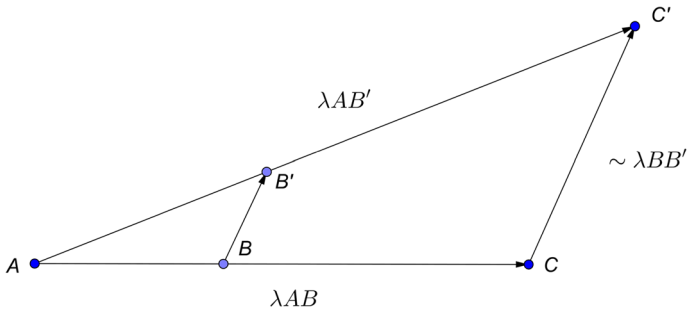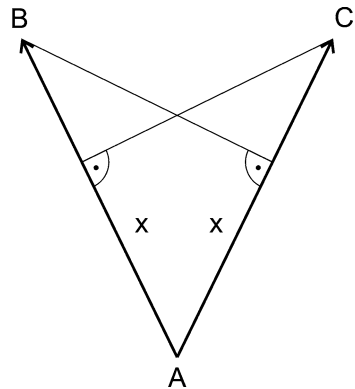the **circle** with center S and radius r is $C(S, r) = \{T : |ST| = r\}$

Fig. 22 Illustration for the scale invariance axiom

Fig. 23 Illustration for the axiom A13

Axiom (A11) *If a line has two common points with a circle, points A and B, then the midpoint P of AB is the point on the line nearest to the center of the circle.*
(by the isotropy principle and an idea of continuity of space)
Axiom (A12) *If a line has exactly one common point with a circle, then the common point is the point on the line nearest to the center of the circle.*
(by the isotropy principle and an idea of continuity of space)

The definitions of perpendicularity of lines, orthogonal projection and scalar orthogonal projection of an arrow:

A line a is **perpendicular** to a line b, in symbols $a \perp b$, if a intersects b and there is a point S on a which is not on b such that the intersection of a and b is the point on b nearest to S.

orthogonal projection of a point S to a line p is the point on line p nearest to the point S.

The **scalar orthogonal projection** of the arrow AB onto the arrow CD, denoted AB_{CD}, is the $\pm$ length of the orthogonal projection of the arrow AB onto the line $p(CD)$, where the sign is $+$ if the projection is in the direction of CD, $-$ otherwise.

Axiom (A13) $|AB| = |AC| \rightarrow AB_{AC} = AC_{AB}$. (Fig. 23)
(by the isotropy principle)

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