MATHEMATICS FOR PRESCHOOLERS

Handbook for parents and educators [1]

Version 4[2]

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The classification of geometric shapes with the aim of increasing mathematical literacy does not have such a creative charge as the colors and shapes on the zebra's butt.

Lavoslav Čaklović

Play is the child's work.

Friedrich Fröbel

1 Preface

The handbook contains brief instructions and examples of mathematical activities. In the INSTRUCTIONS section, instructions are given on how, and in part why that way, to help preschool children in their mathematical development. In the ACTIVITIES section, there are examples of activities through which the child develops her mathematical abilities. If you don't want to read the instructions but just use these activities, you can use them in any order. The only thing that is crucial is to keep the following in mind:

A child's mathematical activities must be part of the child's world - they must have their motivation, meaning and value in the child's world, and not outside, in the world of adults. A child develops his mathematical activities best through play. In developing mathematical abilities, the child must have freedom and not pressure to learn something.

It is up to us to help the child and to guide her in his mathematical development by providing her with the appropriate environment for it: space (either in the house, in the children's park or in nature), time, material for play (puzzles, paper, clay, sand, ...), designed elements (games, toys, books, ...), and motivating context (stories, arrangement of her shelves, participation in family affairs, etc.); and that, respecting her individuality and pace, we offer various activities and unobtrusively support those which attract her at that moment.

Seen in extremes, a child will develop much better mathematically if we let her play in peace, than if we put pressure on her. When we participate in her activities, let's keep in mind, as my wife said, that a child learns best when she doesn't know that she learns.

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[1] This material may be used freely (please, cite the source), except for commercial purposes. Examples are mostly part of „folklore“. For some examples, the authors are known to me, some I changed and some I made up. In the Appendix there are sketches for two stories where I indicated my authorship with the mark ©Boris. Sometimes I will give a reference to commercial material. As one would say, I have no conflict of interest here. All comments and suggestions are welcome. Please send them to boris.culina@vvg.hr.

[2] The material requires editing, especially in English language proofreading and designing illustrations. Every help is welcome.

Contents

1 INSTRUCTIONS

1 INTRODUCTORY PART

1 On the reasons for creating this material

This material is intended for parents, educators and anyone who wants to help children in their mathematical development. The material deals primarily with the mathematical development of preschoolers, but the approach can be applied to older children as well. The material is not standard - it is in opposition to the official educational mathematics standards for that age. That is why I consider it necessary to write how it was created, so that the reader clearly knows what they are getting into when applying this material.

I am a mathematician by profession with many years of experience in higher education.[3] It all started when I felt that it was time to help my then three and a half year old granddaughter Nina with her mathematical development. I immediately remembered the book W. Servais, T. Varga, (Eds.) *Teaching School Mathematics*: Penguin Books – Unesco (1971),[4] that left a deep impression on my teaching of mathematics, especially Varga's introductory article. His words „Every child, by nature, likes learning just as he likes eating.“ were deeply engraved in me.

Too many students I worked with didn't have that hunger to learn math. For most of them, that hunger was replaced by an aversion to mathematics. It is easy to see that it is a mass phenomenon all over the world. I have always considered it an unacceptable and unnecessary state of affairs. However, when I entered the world of teaching mathematics to the youngest and realized that the same situation exists there, moreover, that it arises there, I literally experienced it as violence against children. Not only is this violence institutionalized in the theory and practice of mathematics education, but it is well ingrained in the minds of most people.

It especially bothers me when I hear, and I hear it often, that a child is not good at math. In addition to the fact that we should be very careful in making such judgments (especially since we must not express them in front of children), how can we even say such a thing and unnecessarily disqualify a child, even convince her that she is not for mathematics if we ourselves do not understand what mathematics is? I responsibly claim that not only most people do not understand what mathematics is, but also the established standards of mathematics education for the youngest do not understand what mathematics is.

The established and institutionalized point of view is that mathematics is the science of numbers and Euclid's geometry. Such a point of view was overcome at the end of the 19th century, when mathematics underwent a real revolution and passed into what we now call modern mathematics. After the Second World War, various groups of mathematicians tried

[3] You can find out more about me and what I do on the website https://understandingmath.academy/about-2/ or on Twitter: @doing_math. On this occasion, it may not be out of place to mention that I am the main author of a dozen mathematics textbooks for undergraduates and co-author of mathematics textbooks for the 5th to 8th grade of elementary schools in Croatia.

[4] The book can be found at the following link: https://unesdoc.unesco.org/ark:/48223/pf0000007959 .

to introduce modern mathematics into the education system - this is the so-called new mathematics movement - but they did it unilaterally, without understanding the developmental psychology of children. This had the most negative effect on the education of the youngest. Not only did they fail, but they created a negative attitude towards the introduction of modern mathematics content into education. The mathematics education of the youngest was once again taken over by educators who, I must say, generally speaking, do not understand modern mathematics, just as the creators of the „new mathematics" did not understand educators either.

Although elements of modern mathematics are present in the current standards, they are marginalized - the standards are again dominated by numbers and geometric figures. All other very important and modern mathematical contents are subordinated to them. Thus, by adhering to the current standards, we drastically limit the mathematics of the child's world, hamper the correct mathematical development of a child, and we can turn her away from mathematics.

And all that is unnecessary. Not only is today's real and successful mathematics precisely modern mathematics (which of course includes numbers and geometry), so it is worth acquiring, but modern mathematics is precisely the mathematics that is natural to children's world. This mathematics is presented in these materials, but with a very different approach from the approach in the new mathematics movement.

I am deeply convinced that changes in the mathematics education of the youngest are necessary and I hope that the time for them has come right now. A sufficiently broad understanding of mathematics is very important here. We don't need to worry that we might see mathematics too broadly. This will not limit anyone in mathematical development. We just must not allow ourselves to see it too narrowly, which is what dominates today. The main purpose of this material is to offer sufficiently wide and free activities for the correct mathematical development of the child.

Since I want this material to be helpful to as many people as possible, I will try to be as specific as possible: give instructions on what to do without explaining too much why these instructions are correct. I will put part of the explanation in the footnotes, for those who are looking for a more detailed explanation, ready to face more professional terminology. I will often refer here to parts of my article *Early Years Mathematics: the missing link*[5] (abbreviated EYM in the text) where I explained this approach in more detail. In other words, this material is simply that article transformed into a usable form and supplemented with concrete examples.

[5] The article can be found at the link https://philpapers.org/archive/ULIEYM.pdf. It has been accepted for publication in the next issue (autumn 2023) of the *Philosophy of Mathematics Education Journal.* Underlying the article is the understanding of mathematics described in my article *Mathematics – an imagined tool for rational cognition*. The article can be found at the link https://philpapers.org/archive/CULMA.pdf. The first seven sections have been accepted for publication in the journal *Annals of Mathematics and Philosophy*.

In order to help children in their mathematical development, it is good that we ourselves have some mathematical knowledge. Short and intuitive explanations are given in this material.

2 Why mathematics is important for preschoolers

The preschool period is the most important period in human development, both emotionally and intellectually. This is where the main foundations of personality are created. The creators of modern education for the youngest already knew this clearly, starting with Friedrich Fröbel, the creator of kindergartens in the first half of the 19th century, and Maria Montessori, to mention only those whose works I know best and which are closest to my heart.[6] Modern scientific research confirms this, including the almost decisive importance of that period for a child's mathematical development.

I would like to point out that such importance does not require us to worry and stress whether the child will develop mathematically correctly during that period, but only to provide her with enough space, time and materials for free play, and listen to what she needs. The child's development does not come from us, but from her. We just need to support it.

The latest research has shown that children of preschool age are much more mathematically capable than previously thought. These results did not surprise me at all. It has long been known in developmental psychology that the preschool age is a period of exceptional creativity and imagination, if the child is provided with the appropriate conditions. And in my view of mathematics, creativity and imagination are key elements of mathematical activities, although in public opinion these abilities are usually associated with art.

3 How to teach mathematics to the youngest

The methodology is mostly established (through the works of Fröbel, Montessori and many others) and gives quite satisfactory answers to the question of how to teach children anything, including mathematics.

The child's mathematical activities must be part of the child's world - part of her daily activities, part of her play, incorporated into children's stories that she likes to listen to. Mathematical activities must have their motivation, meaning and value in the world of the child, not outside, in the world of adults. In developing mathematical abilities, children must have freedom and not pressure to achieve certain learning outcomes.

Just as Georg Cantor (1845 – 1918), the creator of set theory and one of the creators of modern mathematics, said that the essence of mathematics is its freedom, so the essence of

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[6] Studying their works, I was personally fascinated by the wealth of educational knowledge they left behind and at the same time frustrated by the ignoring of this knowledge in today's wider educational practice. You can find their works on the website https://archive.org/ (type their names into the search field).

a child's mathematical education is his freedom. *It is up to us to help and guide children in developing their mathematical abilities, respecting their world and their individuality - which activities and at what point in their development attract them - as well as providing them with the appropriate social environment for this.*

It is crucial that you stick to this simple methodology that only requires love and feeling for the child. Working in accordance with it gives exceptional results. Its violation can have negative consequences for the child's mathematical development.

Educators and teachers must be especially aware of this when they are exposed to the demands of superior authorities that are often in conflict with this methodology. But also parents, when they are under pressure that their child should achieve the required results. Developing such awareness is especially important because the existing school systems are mostly contrary to this methodology, in theory with their uniformity and evaluation system, and in practice with the difficult working conditions of educators and teachers.

4 What mathematics to teach the youngest

First, we need to have a correct idea of what mathematics is. It is something much broader than numbers and geometry.

Mathematics is the human activity of building various ideas whose purpose is to serve us as means for understanding and controlling reality, and for organizing our own activities. Mathematics is our thinking tool for cognition of the world, not the truth about the world.

Numbers are just one, albeit the oldest and still most important, such idea. Using natural numbers and the counting process, we control multitudes. Using real numbers and the idea of measurement, we survey the world. Not only can we precisely determine something with the corresponding number by measuring, but in this way, we can express the laws of nature precisely in terms of the relationships between the corresponding numbers. But mathematics does not stop at numbers. It is something much broader, and numbers are one of its jewels. *The goal of mathematical education of the youngest is not to polish that jewel but to acquire the very heart of mathematics that creates such jewels.*

The idea of space (geometry) in which we move is another such important idea. With it, we control our position in space, possible directions and amounts of movement, as well as various designs and constructions. It helps us to orient ourselves in space and to register and control spatial relationships in various phenomena.

However, modern mathematics has created a whole menagerie of ideas, from simple ideas about sets of objects, relations and operations between these objects, to complex mathematical structures. These ideas help us to describe what happens in small parts of space and in large interstellar spaces, to design complex technological processes and construct modern devices, but also to organize other mathematical structures. These ideas are embedded in almost every more complex human creation, from concrete devices to powerful scientific theories about the world.

Despite the great variety of modern mathematics, its basis is simple - it springs from our internal world of activities. These are activities over which we have a distinct control and which we organize and design according to our own measure, in our everyday and subordinate world. These are, for example, movements in a safe space, grouping, arranging and connecting small objects, spatial constructions and deconstructions with small objects, talking, writing and drawing on paper, shaping and transforming manipulative material (clay, paper, ...), making choices , combining and repeating actions, dynamics of actions and changes in the environment subordinate to us, painting, singing, ... *Internal activities are unique activities of the human species, our evolutionary gift, which are only present in traces in other animals*. They are part of our innate abilities that, unlike other animals that adapt to the environment, we adapt the environment to us. *It can be argued that mathematical ideas arise through designing, conceptualizing, idealizing or generalizing happenings in our internal world of activity.*[7] Of course, cultural evolution and social context play a key role in designing and conceptualizing these activities.

It is easy to see that all the described activities are very present in children, moreover they form the very basis of children's play. During growing up, these activities develop, but this is primarily the development of their design and conceptualization (we organize these activities according to certain ideas, such as, for example, the process of counting objects). Thus, *children's play is the basic framework for a child's mathematical development*.[8]

Often the purpose of children's play is to understand the outside world, for example when they play doctor or cook. When the play takes on such a purpose, then in the children's world we have a mathematical model of the situation that children process through the play. The lesson is clear: *the more play, the more math in the children's world*.

Children’s stories themselves can be understood as mathematical models of certain phenomena. The Witch, for example, represents evil, Hansel and Gretel goodness, which, aided by wisdom, defeats evil and forgives the deceived (their father) but not the incorrigibly evil (the Witch and their stepmother). Here art and mathematics are almost indistinguishable.

Both art and mathematics, these unique creations of the human race, have the same source: in the internal world of children's activities. Both of them are characterized by creativity and imagination. They are later separated by purpose: whether emotions or intellect are more pronounced in shaping internal activities. Unfortunately, „thanks“ to the education system, they are also separated by the attitude of the children towards them. As a child grows up, generally speaking, art and math take on opposite values: art remains attractive and mathematics becomes repulsive. The main reason for this is that in educational systems the attitude towards art is free and optional, while the attitude towards mathematics is

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[7] You can read more about what our internal world of activity is and how mathematics arises from it in EYM on pages 5-8.

[8] All children's activities are best developed through play. Fröbel recognized this most clearly a long time ago: „Play is the highest expression of the child's human development, it is the free expression of what is in the child's soul.“. This is especially important for mathematics and art because internal activities are the very core of play, and mathematics and art are developed from these activities.

burdened by a narrow view of mathematics and the repression of „educational achievements“. This is what must change. *If in education the attitude towards mathematics was as free and optional as towards art, mathematics would remain as attractive to children as art. And developing such a positive attitude in children towards mathematics is far more important than any specific mathematical knowledge.*

In addition to play, children develop mathematics whenever they try to organize their daily life with the help of adults: when they arrange their clothes in drawers, plan activities, etc.

In short: *children's mathematics consists of the world of children's internal activities that they eventually purposefully organize, design and conceptualize in order to understand and control the external world and organize their actions in it.*

This view of mathematics education for the youngest is in complete harmony with the above-described methodology for teaching the youngest. It naturally extends the approach of Fröbel, Montessori and others to the field of mathematics.

It is up to us to

1. provide an environment for children in which they can spontaneously develop their activities:
    a. suitable space and time: room, children's parks, beach, forest, …;
    b. manipulative material: clay, sand, leaves, …;
    c. designed elements: games, toys, books, …),

2. to create the appropriate motivational context: stories, arranging the child's things, the child's participation in family affairs, participation in activities with other children,

3. and to unobtrusively teach them in those mathematical activities in which they currently show interest. For example, if a child shows an interest in folding and cutting paper, let's show her how to make an airplane by folding it, or how to make paper snowflakes by cutting folded paper. We just have to be careful here. If the child refuses or shows disinterest in what we show her, don't force it. Maybe tomorrow or in a month she will ask: „And how do you make a paper plane?“[9]

Creating a motivational context is particularly important. Stories and imaginary plays are ideal for creating a spiritual environment in which the child will naturally show interest in certain activities. For example, Darko Ban, my Croatian variant of Peter Pan, helped me a lot. Darko Ban is an astronaut who flies around space in a rocket, and when he has a problem, he flies to Nina’s (my granddaughter) house at night, leaves the problem on her table with a request that she help him solve it. Nina solved these problems with great will. Out of gratitude, Darko Ban would sometimes bring her a gift from his travels, put it in a secret place in the house, and leave her a map on which the secret place was drawn. This way, Nina not only received gifts, but also learned to orient herself on maps. In the end, she grew to

[9] I learned geometry on a deeper level not in school but playing football. On the other hand, I'm still extremely bad with knots to this day. I have no feeling for them at all, because I never „worked“ with knots as a child.

love solving problems so much that Darko Ban did not need to ask her to handle the problems anymore (but he could continue to bring gifts).

Likewise, it is very convenient to place problems in a story as problems that the heroes of that story must solve with the help of the child in order for the story to have a happy ending.

In the same way, *concrete jobs - making food, working in the garden, cleaning the room, making a birdhouse out of wood, etc. - are also ideal environments in which various mathematical content naturally emerges.* Thus, in addition to the physical environment, we need to create a spiritual environment (context) for the child's mathematical activities. It gives the child the necessary motivation for certain mathematical activities.

Fortunately, many of these mathematical activities are developed by the child through other content that is widely believed to have nothing to do with mathematics: fine arts, literature, physical culture, socializing through play, organizing activities, taking care of order and tidiness through, for example, kindergarten socializing or participation in household chores. The problem is that the child here can show more pronounced mathematical abilities, and the adults „declare“ her that she is not good at math because she does not do very well with numbers, probably only because they are not interesting to her at the time.

For me personally, mathematics was totally uninteresting until the sixth grade - just some math drill that didn't go well for me because I mixed up the digits 5 and 7 in writing. In the sixth grade, I got a new math teacher, Nevenka Bogdanić. She showed us how equations can be used to easily solve a problem that I thought was impossible to solve. That's when I experienced a „click“ for math. I mention this to emphasize that we need to give the child time for some things to become interesting to her. Of course, this example also shows the importance of correctly helping the child in her development. This example carries another lesson: it is very important to expose the child to various activities. In this way, she will best discover what attracts her (experience the „clicks“) and develop in her natural way.

In the next, more practical part of this material, more distinct mathematical elements will be highlighted. They are more distinct in the sense that they train children for a more distinct control of reality. But let's not lose awareness and orientation that children's mathematics encompasses much more than these isolated elements. Basically, it is free play and storytelling, as well as the organization of the child's daily life.[10]

In what follows, I will distinguish direct mathematical elements (such as numbers or orientation in space) and background mathematical elements – elements that are present in all mathematical activities (such as language and procedural thinking).

In the ACTIVITIES section, examples of specific activities are sorted by mathematical elements that I will list in the remaining part of the instructions, but this order is not important for their use. You can offer any of them to the child at any time, regardless of other examples.

[10] A critique of the mathematics education standards on this issue can be read in EYM on pages 10-19.

2 DIRECT MATHEMATICAL ELEMENTS

1 Sets, relations, and functions

Modern mathematics is like Lego bricks. The basic material from which its structures are built are sets, relations and functions. Although they are basic material, they are usually less well known than other mathematical elements, for example numbers and geometry. So I will dwell a little more on their description here than with other elements.

When we consider some objects, we naturally consider sets (multitudes, collections) of these objects, for example, a set (group) of children playing in a park. The mathematical idea of *set* generalizes this situation. Whenever we somehow manage to „separate“ some objects, we say that we have a set to which these objects belong, while other objects do not belong to it. So, for example, we have a set of all the characters in a fairy tale (what belongs to the set is clear, although these characters do not exist in reality), the set of the child's ancestors (what belongs to the set is clear, although we cannot effectively determine this for a person from the past), but also the set of children in kindergarten over 10 years old (no child belongs to this set - it is an empty set).

Sets are *equal* when the same objects *belong* to them (these objects are called *elements* of the set). For example, the set of children in kindergarten who are over 10 years old is equal to the set of all the people who have been to Jupiter. We just described the same set - the empty set - in different ways.

We can take part of the objects from the given set, arbitrarily or by some selective property. Thus we get a *subset* of the given set. We can take common elements from two sets and make a new set from them, the so-called *intersection* of those sets. Or we can take the objects from both sets together: thus we get a new set that we call the *union* of these sets. In the same way, we can take all the elements that are outside of a set (and that are inside some total set of objects that we are considering): thus we get a new set that is called the *complement* of the given set.

When we consider some objects, relations between these objects naturally arise in our consideration, for example, that two children are friends, that Ermin is taller than Azra, and that Antonio is the grandfather of Marino. The mathematical idea of *relation* generalizes such situations. Whenever we somehow manage to separate some pairs of objects, we say that we have obtained a relation: the first object in the pair is in that relation with the second object in the pair. The order of members in a pair is important here (that's why the term ordered pair is used in mathematics). For example if Antonio is Marino's grandfather, then Marino is not Antonio's grandfather. Sometimes we can exchange places in the relation (then we say that the relation is *symmetric*). For example in the relationship *x is a cousin of y* we can do that. If Ivan is a cousin of Marina, then Marina is also a cousin of Ivan. When the relationship is symmetrical, we don't have to worry about the order of the words. For example here we simply say that Ivan and Marina are cousins.

As with sets, we consider relations to be *equal* if the same pairs are in those relations, regardless of how we describe the relations.

Relations do not have to connect two objects (so-called binary relation), but several objects. For example the relation *the number x is between the numbers y and z* connects three objects (the so-called ternary relation), but binary relations are the most common.

When we consider some objects, functions (assignments, mappings, transformations, operations) between these objects naturally appear in our consideration. For example, we can assign to each person *x* their mother: *the mother of x*. Here, the connection of this function with the relation *y is the mother of x* is immediately visible. *Functions* are a special type of relations where pairs are connected in a specific way: each *x* is paired with only one *y* - each person has only one mom. This uniqueness of the second member in the pair means that *y* is determined by *x* (for a given *x*, the description *the mother of x* identifies exactly one person. As opposed to that, with the relation *y is a friend of x* we do not have this unambiguity because *x* can have several friends. So we cannot unambiguously identify a friend to *x*, because there are more of them. Likewise, the description *the nose of the person x* determines a function (we assign to the person *x* their nose), while the description *an ear of the person x* does not describe a function, because a person has two ears: we do not have a precise determination of which ear we assign to them.

We usually describe a function by specifying a unequivocal rule by which we assign to an object *x* an unambiguously determined object *y*. Thus, while the rule *an ear of the person x* does not describe a function, the rule *the left ear of the person x* describes a function because now it is precisely determined what we assign to a person. If by any chance a person does not have a left ear, then in neutral mathematical language we say that this function is not defined on them.

We consider functions *equal* when they connect the same objects regardless of how we describe them. For example the function given by the rule *go two steps forward, turn 90 degrees to the right and go three steps forward* assigns to each body position (place and orientation of the body at that place) the same position as the function given by the rule *turn right 90 degrees, go three step forward, turn left 90 degrees*, *take two steps forward and turn right 90 degrees*. It is the same function, only described (given) in two different ways.

A function can assign an object to a pair of objects, rather than to a single object. Such is, for example, the operation (function) of addition, which, for example, assigns the number 8 to the pair of numbers 3 and 5. A function can act on entire set of objects. For example the function *the largest number in a set* assigns to each set of natural numbers the largest number in the set.

Sets, relations and functions, the basic elements from which modern mathematics is built, appear in the children's world as basic children's activities from which they assemble more complex activities. Sets correspond to the grouping of objects, relations to connecting objects, and functions to actions on objects. In helping the child to develop these activities, it is important to keep the following in mind. While the objects that children group into sets, examine the relations between them, or assign one to another, are concrete objects, sets, relations and functions have no concrete characteristics. We cannot just point a finger at them. Because of their abstractness, these terms should be avoided when working with children. Let's not talk to them about sets, relations and functions, but let's support them in grouping objects, connecting objects and actions on objects: let's support them in „building" concrete sets, relations and functions. We will talk about the toys on the table and not about the set of toys on the table, that Nina and Ezra are cousins and not that they are in the relationship *being cousin*, that Anja is Ezra's mother and not that the function *mother of* assigns to Ezra his mother Anja.

Since sets, relations and functions occur in all situations, they are background mathematical elements. However, children build concrete sets, relations and functions. Thus, in the children's world they primarily appear as direct elements and not as background elements. Given that, as the above examples testify, everything around us is „bustling" with sets, relations and functions. There is plenty of material for the „construction" of concrete sets, relations and functions.[11]

In the ACTIVITIES section, a whole host of activities with sets, relations and functions that „populate" the children's world are displayed.

2 Structures

We build mathematical *structures* from sets, relations and functions. Just as with Lego bricks we build more complex constructions from basic elements, so here too, we simply take a couple of sets, highlight some elements in them, add some relations and functions between them, and look at it all as a new whole.

For example, we can single out a set of people with the relationship between them *being a neighbor*. A structure consisting of one set with some relation on that set is called a *relational structure*. In the part with relations, we were constantly looking at relations over some set, so we were constantly working with relational structures.

If we select some operations (functions) over a set, then we are talking about an *algebraic structure*. In the part with functions, we constantly looked at functions not by themselves, but how they act on some sets, so we actually worked with algebraic structures.

We build more complex structures from simpler structures. One such structure is the structure of real numbers. It consists of a set of real numbers, among which we have two prominent elements zero and one, four operations - addition, subtraction, multiplication and division - and a relation of comparison between numbers.

In the children's world, the construction of structures from sets, relations and functions occurs in the organization of basic activities into more complex activities that have some integrity and purpose. Such are, for example, various games, but also the organization of things (e.g. decorating a room) and planning some activity (e.g. the order of visiting the animals in the zoo).

[11] A more detailed discussion of sets, relations and functions in the children's world is presented in EYM on pages 12-14.

In the ACTIVITIES section, some activities are listed that are closely related to simple mathematical structures.

3 Natural numbers

In the standards of mathematics education, as well as in what is commercially offered on the market, too much attention is given to numbers, hard vocabulary is used and things are set almost fatefully – „If the child does not now acquire a deep understanding of numbers and operations with them, she will be mathematically ruined forever.“. I have to say right away that this is totally wrong. Unnecessary pressure is placed on both parents and educators, and worst of all, on the children. As my daughter said: „Why did they push so hard with those numbers. Everyone learns to count and calculate, sooner or later.“ This pressure is especially present because the algorithms for performing operations with numbers are forced too early. At that age, these algorithms are too artificial for children, if they are done formally, and too complex, if we try to explain them in a meaningful way.

Numbers are the oldest and still the most significant mathematical structure, but they are not a bugbear nor are they a measure of a child's mathematical development. They need to grow slowly in the children's world, perhaps for years. Given that they can be „boring“, we should take care that they develop naturally in the children's world, and not impose them as learning outcomes that the child must achieve. With such a fateful approach and demands to achieve planned outcomes, we only frustrate the child and distance her from mathematics. Reducing math to numbers and forcing children's achievement with numbers is the main reason why we have so many people who don't like math and think they are incapable of math.[12]

Natural numbers arise from activities related to the comparison of sets of objects. By connecting objects from one set with objects from another set so that one object from one set is connected to at most one object from another set (we establish the so-called 1-1 correspondence between objects from two sets), we can determine which set has more objects or whether the sets have the same amount of objects. For example if the children in the park come to the play yard with swings, each child will look for a swing for themself. That way, they will quickly find out if there are the same number of children and swings, or if there are too many swings, or if a child will be left without a swing. Thus, the first and basic thing is to help the child in developing this so-called *1-1 comparison* of sets. With such a procedure, children can easily find out where there are more and where there are fewer objects, and when they are equal.

The next step is the acquisition of the *counting process and natural numbers* as objects with which we count. Numbers serve to represent (measure) the size of a set, and counting is the process of measuring the size of a set. By counting objects in a set, we establish a 1-1 correspondence with the initial sequence of numbers. The number at which the count stops

[12] You can read about the unnecessary and dangerous pressure that mathematics education standards place on the acquisition of numbers in EYM on pages 11-12.

is the measure of the set. For example if we counted that there were three objects in a set, we established a 1-1 connection between those objects and the initial sequence of numbers: 1,2,3. The last number in that sequence, number 3, is the measure of that set. Now, instead of comparing the sets directly, we compare the numbers associated with them. For example if one set has three objects and the other has five, given that five appears after three in the sequence of numbers, we know based on these measures that the second set has more objects.

So what are natural numbers? This seemingly difficult question is basically an irrelevant one. The objects we choose as the standard with which we count are called natural numbers. If we count using pebbles, then these are numbers. For counting, it is only important that we have the first number from which we start counting, we call it one, and that each number has a next number (new compared to all previous numbers) with which we can continue counting, if necessary. Conceptually, there is no reason to single out any special objects for this purpose. Practically, for the purposes of counting and calculation, we choose a suitable standard, in the past beads on an abacus, today strings of decimal digits on paper or bits in a computer. The idea of natural numbers is like the idea of chess. Just as we can play chess with wooden figures or with characters on a computer screen, or by correspondence, we can also choose various standards for numbers. Just as we can always transfer the game of chess from one realization to another, we can always transfer counting and calculation from one standard to another.

So what are natural numbers in the children's world? These are simply spoken words: one, two, three, ... because it is the simplest and most accessible series of objects suitable for counting. This is exactly what we use when we count, and children easily learn it from us. The time order of pronunciation naturally determines what the first number is - the first spoken word - and after each spoken word we have the next spoken word in time - the next number. In the Croatian language, it is a series of words: jedan, dva, tri, ... When children in Croatia learn to count in English: one, two, three, ..., they just switched to another counting standard. In mathematics education standards, as well as in the commercial market, numbers are presented to children as written symbols - numerals: 1, 2, 3, ..., 12, ... Numerals are better avoided or not given importance at the initial level of learning numbers for several reasons. The main reason is that they do not have the natural order that spoken words have in a time series, which is essential for the counting process. Furthermore, they require certain reading and writing skills from children. Reading and writing is a complex process, which children have not yet automated, and which unnecessarily burdens the mathematical content. For this reason, writing and reading in mathematical activities at this age should generally be avoided or minimized, or introduce as much as the children are currently ready for it. Finally, numerals are symbolic records (we do not read them as ordinary written words) that introduce an unnecessary element of abstraction into the counting process.

Among the natural numbers, we can insert a new number at the beginning, which we call *zero* or *null*. We can understand zero as a number that indicates the initial state of counting, the state before we started counting. If there are no objects in the set we are trying to count then we cannot continue counting. We are left with the number zero, so it is a measure of the empty set. For example, we want to see how many cakes are left in the fridge, and it turns out that they have all been eaten: there are zero cakes in the fridge. Because of the surprise that usually accompanies the discovery that something is not there, children easily accept zero (null: from the Latin nihil = nothing).

By counting, children can determine in which set there are more objects by comparing the numbers that they assigned to those sets in the counting process - which number appears before and which after in the time series of numbers (as I mentioned, the word set itself should be avoided or at least not highlighted).

Children directly understand which number is the immediate successor of a given number (the next one in the time sequence, e.g. after five comes six) and the immediate predecessor (the previous one in the time sequence, e.g. before six is five). However, it is more difficult for them to determine the predecessor because they have to reverse the time order in their head, go back to the immediate past, in order to determine the predecessor of the given number.

Also, children can perform simpler operations with numbers - addition and subtraction. But it should be done verbally and substantively. Not on paper, because writing and reading, as I have already stated, unnecessarily burdens the mathematical content, which thus loses its clarity. And not formally (when something is done only with form), with some procedure with digits or voices, but substantively, because formal procedures are mostly foreign to children. Content-wise, addition represents the union of disjoint sets, and subtraction represents the „extraction“ of one set from another. So, for example, to add four and five, they need to collect four objects, then five new objects and then count how many of them there are together (the sum is nine), and to subtract four from five they need to collect five objects and remove four objects from them (difference is one).

If the child shows an interest in formal addition and subtraction, below in the activities with numbers there is a suggestion for how she can do this. (page 59).

I think it is enough to stick to this in preschool age, not to go to excessively large numbers and to other operations (e.g. multiplication and division), unless the child shows an interest in such a thing, or it appears naturally in the context.[13]

4 Geometry

Children's space is primarily a space for their movements, construction and deconstruction in space, and navigation in space. Therefore, in their geometric development, the emphasis should be on the development of these activities. They develop this best on playgrounds, in sports activities, orienteering in nature and making various constructions with „cubes“ (wooden bodies, Lego blocks, magnetic characters...), clay, paper, etc.

[13] It never crossed my mind to do multiplication and division with Nina, but while making cardboard cars (see the example on page 47) we naturally got into the situation of multiplying and dividing some numbers.

To that should be added the development of a visual representation of problems, with the help of which the child creates mathematical models for different situations. For example, ordinary drawing of an elephant is the creation of one such model in which, in addition to creating a model of an elephant (which improves more and more over time), the child also develops a sense of spatial relationships. Therefore, in geometric development, the child should be allowed to draw or model as much as possible in clay, dough, etc. This also includes reading and making spatial maps of the room, the environment in which she lives, the park where she plays, etc.[14]

Considering that in the standards of mathematics education (as well as in the commercial market of children's products), geometric figures and solids (triangles, quadrilaterals, ...cubes, ...) and their properties are too much emphasized, I would like to point out here that they should not be given so much attention . At this level, for children these figures are mostly just elements for constructions and deconstructions, and they acquire them through these activities. The figures themselves do not particularly interest them outside of these activities, unless they have some attractive symmetries. Many think that children have mastered geometry when they master the distinction and properties of these figures. It's not like that. The activities described in the previous two paragraphs are the most important for their geometric development.

5 Other direct mathematical elements

Here I would like to point out some more mathematical ideas that naturally spring from the children's world, and which are represented in the ACTIVITIES section, but not as separate topics but as activities within „*bigger*“ topics (structures, geometry, procedural thinking...). These are the ideas of **recursion** (initial elements plus rules of construction - this is often present in games or geometric constructions), of **states and transitions** (this is present in most games - we usually show this idea with graphs where states are points and transitions are arrows connecting the points), of **transformations** (this is present whenever we change something according to some rules - a kind of realization of the concept of function), of **analysis and synthesis** (*disassemble, analyze, assemble* - it occurs everywhere), of *symmetry* (it can develop quite nicely within geometric activities), **coordinatization** (objects are marked (represented) with some signs , in a more developed form, usually numbers), of **closeness and approximation** (can be developed within geometric and numerical activities), of **chance** (a typical example is throwing a dice in a game or in fair distribution, e.g. determining who will start the game), of **choice** (when guided with some goal, we make a choice that we think leads us closer to the goal), of **repetitions** (we repeat something until the goal is reached), of **changes** (something has increased or decreased, something is faster, something is slower), etc.

[14] You can read more about geometric activities in EYM on pages 15-16.

3 BACKGROUND MATHEMATICAL ELEMENTS

Background mathematical elements are elements that are present in all mathematical activities.

Such are sets, relations and functions, which are placed here under direct mathematical elements because they are mainly present in the children's world in that way. However, we should encourage children to use these elements in all their activities: to notice certain sets, relations and functions relevant to the given situation.

Background mathematical activities also include developing the ability to abstract and represent problems, procedural activities, logic and language. However, in my opinion, the most important thing is language, especially since it encompasses logic and the process of abstraction. The importance of representation has already been discussed in the variant of geometric representation. Likewise, as mentioned in the introductory part, when children play some situation (doctor, policeman, ...), they represent certain life situations in such a play. This will be further illustrated below in the problem solving section. Also, certain situations are represented through children's stories, as will be illustrated below in the section on language.

1 Language

Language is perhaps man's most important tool for controlling reality.[15] Not only do we use language to transmit knowledge, but we also use language to shape knowledge. That is why language is a very important part of mathematics. It is present in children's activities from the beginning and develops best through these activities. Unlike me, most do not consider that in one of its essential aspects language is part of mathematics. But regardless of whether you consider language a part of mathematics or not, a child's language development should be given a lot of attention.

Language is a very powerful mathematics. By choosing words in a certain situation, we make an abstraction, we extract from the situation what we are interested in and abstract the rest. It is our key mechanism by which we deal with the complexity of the world. For example, when we analyze a football match with words like offside, dribble, double pass, ..., we select what is important for the game and focus on that, while abstracting the referee's hairstyle, the bird that flew over the stadium, etc.

Furthermore, we use words to structure and control those aspects of the situation that interest us. Let's put ourselves in the role of those who once upon a time invented the football game. They had to introduce all these words into the language, give them meaning and thus turn the uncontrolled race for the ball into a structured game. Nouns give us

[15] *If someone wants to know more about the importance of language for human rational cognition, they can look at my article The language essence of rational cognition with some philosophical consequences* published in the journal *Tesis*: https://philpapers.org/archive/CULTLE.pdf.

control over objects, and predicate expressions over actions. Moreover, we use predicate expressions to shape and clarify concepts.

Thus, language is an important mathematical element that should be developed from early childhood. Like us, the child understands and controls reality using language (she can express that she is hungry, she understands when her mother tells her that she will come home soon, etc.). Just as the creators of the football game used language to design running after the ball into a football game, so also a child uses language to design and conceptualize her activities, structuring them into designed wholes. She will make a house out of Lego bricks (for that she needs the concept of a house), with the help of adults she will turn the vocal sequence of numbers into a powerful mechanism for counting elements in a multitude, etc.

That's why we help a child a lot in mathematical development whenever we read her stories, listen to her when she tells us stories, describes various situations and expresses her thoughtsand feelings, and when we encourage her to communicate with other children and adults.

By encouraging the child to use language in mathematical activities, as well as in other activities, we help her to make abstractions and clarify concepts and word meanings. For example, expressing with language when composing a tangram from geometric figures helps the child to better abstract the irrelevant elements of the figures (type of material, thickness of parts) and single out the essential elements (shape and dimension) for problem solving. At the same time, she acquires, for example, the concept of a triangle and begins to distinguish between the properties of a triangle and the properties of a square. In short, by refining language in mathematical activities, the child refines its mathematics.

Language is used to build children's stories, which are basically mathematical models of various events (in the introductory part on page 7, I explained this using the example of the story of Hansel and Gretel). Through stories, the child acquires an understanding of these events. When she starts to make up stories herself, she starts to create mathematical models with which she tries to understand something. These are all very important mathematical activities.

In addition to the above, language opens the way for the child towards the idealized mathematical worlds that arise from her activities. Thus, for example, through activities with numbers and through the concepts she developed there, she will slowly create an image of the idealized world of numbers: she will talk about *all* numbers and understand what it means that there are an infinitely many numbers. In geometric activities, she will slowly develop the idealized concepts of point, line and plane. At one stage of development she will start talking about parallel lines that never intersect, while the others intersect at exactly one point, etc. It is not a problem for the child to imagine idealized mathematical worlds (not she must imagine them correctly - that comes over time, as part of learning). Just as a child uses language to describe the story of Snow White and the Seven Dwarfs, she also uses language to describe the world of all numbers.

I would also like to point out that at the preschool level, children use language in their activities, but speaking generally language itself is not the subject of their activities. For example, a child will use the predicate expression *to be blue* in sorting objects, but will not analyze what *to be blue* means. The role of language in their world is the same as the role of sets, relations and functions. The same happens with all background mathematical elements. Children use them as a tool in activities, but, generally speaking, they don't think about them. As they use pencils to draw without keeping their attention on the pencils themselves.[16] Thinking about the background mathematical elements comes later.

2 Logic

No matter how you look at logic, it always manifests itself as the logic of language - in connecting the truth values of sentences: what does knowing the truth or falsity of some sentences tell us about the truth or falsity of other sentences. Thus, by acquiring language, children also acquire logic.

Through the use of language, children acquire the meaning of the connectives *is not*, *and*, *or*, *if ... then*, etc. First, they learn the negation („I won't go to kindergarten“) and the conditional („If you let me watch the cartoon, my knee will stop hurting.“). They also quickly acquire the quantifiers *for each* (*all*: „All the dolls are here“) and *there is* (*some*: „Someone is in the basement“).

Children are full of imagination and creativity, and in their imagination and thinking they have a pronounced logic. Logical reasoning works very well for them[17] and they are excellent at detecting inconsistencies in the story, that is, they take care that the story is consistent.[18]

By encouraging children to retell or invent stories themselves and to retell events, to discuss stories and events with other children or adults, to look for reasons for certain actions and events, and to draw conclusions from available information, we best help children develop logic.[19]

3 Procedural thinking

Procedural thinking (how to achieve something) is more appropriate to the dynamics of children's world than declarative thinking (what is and what is not).

[16] The standards of mathematics education for the youngest hardly recognize the mathematical richness of the language and its importance in the mathematical development of children..

[17] Grandmother: “ Santa Claus brings presents only to good children.”, Nina: “ Then Ezra won't get a present.”, Grandmother: “Why?”, Nina: “ Because he wasn't good: he hit me.”

[18] Me: “ What is your doll's name?”, Nina: “Aurora”, Me: “ Didn't you tell me yesterday that her name is Julia?”, Nina: “ Yes, but she changes her name all the time.”

[19] Mathematics education standards do not recognize the richness of children's logic at all.

Procedures must be substantive and not formal and must, at least in the initial phase, be expressed in spoken and visual language and not in written language. Procedures are gradually being specified in the children's world.

The transition to formal procedures, such as algorithms for numerical operations, is a required transition, because formal procedures lose content and include reading and writing skills that children at that age do not yet have. I have already mentioned in the section on numbers why the emphasis must be on content and speech. A formal approach (working with form) is foreign to children at that age, while reading and writing burdens the mathematical content because the children are not yet skilled enough in reading and writing. The refinement of procedures should be gradual, because while precision is gained, freedom of action is lost. Finally, we adults also do not like detailed instructions, but rather more general ones that leave us enough space for independent action. This is even more pronounced with children.

Procedural thinking includes making choices ("I will choose this pan"), composing actions ("I will pour water first, then flour") and structuring actions through repeating a series of actions until some condition is met, (eg "mix the mixture until hardens").

Procedural thinking is best developed through children's rhymes (e.g. *Daddy finger, daddy's finger, where are you?*), appropriate stories (e.g. *The Enormous Turnip* folktale, *A Hen is Going to the Fair* South Slavic folktale, see page **Error! Bookmark not defined.**) songs (*If You Happy and You Know It Clap Your Hands*), spatial instructions (e.g. "go straight to the seat, then right , ..."), cooking recipes, etc.

The development of this procedural component of thinking is especially important today due to the increasing importance of software in modern society.

Simple programming languages such as *Scratch* (https://scratch.mit.edu/) have also appeared, in which children can design characters and program stories in a picturesque environment. However, I am personally reticent about when and to what extent to add computers to children's world. I prefer to avoid it for preschoolers. Nature is (still) their natural environment.[20]

4 Problem solving

Problem solving is not specifically a mathematical activity, but here it is placed in the background activities because it is the ubiquitous natural environment for the development of all mathematical activities.

An essential component of mathematics is that it has a purpose: it is the thought tool of our rational cognition and rational activities in general. This is true for children as well as adults. The only thing is that the purpose of the child's mathematical activities must be integrated into his world, and not into our adult world, which is often mistaken.

[20] The standards of mathematics education for the youngest do not recognize substantive procedural thinking.

Just as the entire human civilization has developed mathematics as a tool for solving large and small problems, and as individuals and groups develop it, in the same way children develop mathematics best by solving problems from their childhood world. As in the world of adults, so in the world of children, this purpose gives mathematical activities integrity - a natural environment for their development. That is why we must keep this component in mind when we help children in their mathematical development. Stories and plays are an ideal environment for setting problems and motivating the child to solve them. These are also problems that arise in the daily organization of life (dressing, arranging toys on shelves, drawers and boxes, etc). Counting on its own can be fun, but it only gets real meaning when counting controls whether all the teddy bears are present at the morning review of stuffed animals.

2 ACTIVITIES

Most of the activities have not been fully developed, but I hope that sufficiently clear instructions have been given for the implementation of the activities. In addition, it is up to you to create physical and motivational environments for some of the activities. So, *these activities also ask you to be active* (which is good). Before the specific ones, here are some „general“ activities.

A1: **Creating a physical environment**. I would like to point out once again that the most important thing is to enable the child to play freely, with possible your help, by creating a suitable space in the house, taking the child to the park or nature, organizing socializing with other children, acquiring suitable manipulative material („cubes“, paper, crayons, clay, etc), … Some even go so far as to consider that it is best at that age to just let the child play. This is certainly much better than putting pressure on the child, but it is still best to unobtrusively help the child develop through this play.

A2: Creating a motivational environment. In addition to creating a physical environment, it is even more important to create a motivational environment in which the child will naturally initiate some activities, such as building houses out of blocks, sketching a map of his room, arranging goods on the shelves, etc. In addition to these instructions, it is good to get books that offer various thinking activities: labyrinths, searching for the right path, marking characters that fulfill a certain property, connecting characters, continuing a sequence, looking for differences in drawings, etc. Those books are full of various mathematics. However, they can be burdened with numbers, especially with numerals (written symbols for numbers) in which the naturalness of counting is lost - there is no natural sequence of numerals similar to time sequence for spoken words for numbers. And that can burden the child (see page 13**Error! Bookmark not defined.**). Be careful with such number problems: it is better to skip them than to burden the child. When using such books, it is good to invent a story in which the child naturally needs to solve such tasks. Darko Ban helped me here (see page 8). Once the child accepts such tasks, she herself will ask you to get her more.

It is convenient to try to realize the tasks from books in a real space with real objects, whenever possible. Physical activities are basic for the child. Pictures carry a level of abstraction that is also good to develop, because they are (but are) a step further in mathematical development compared to real activities. It's the same with other math activities: whenever you can, try to transfer math activities from „painting“ on paper to activities with real objects.

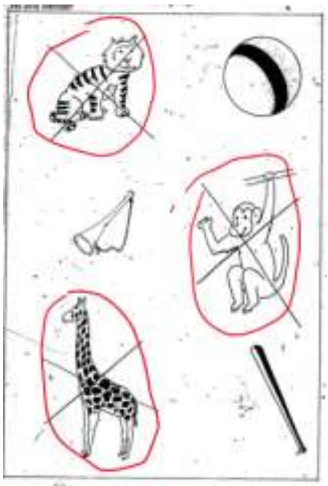
1 Sets

It is better not to mention the word „set“ itself, or at least not emphasize it. A short explanation of the term set and operations with sets is on page 10.

A1: Whether something belongs to a set or not. At the basic level, these are questions like „Is there a teddy bear among the stuffed animals in your room?“, „Is there a sweater with a snowman in your drawer", „Is our car in this parking lot", etc. Speaking more abstractly, when a child determines whether some object has some property (is the car blue, is the cube wooden, etc.), she develops the understanding of various properties.

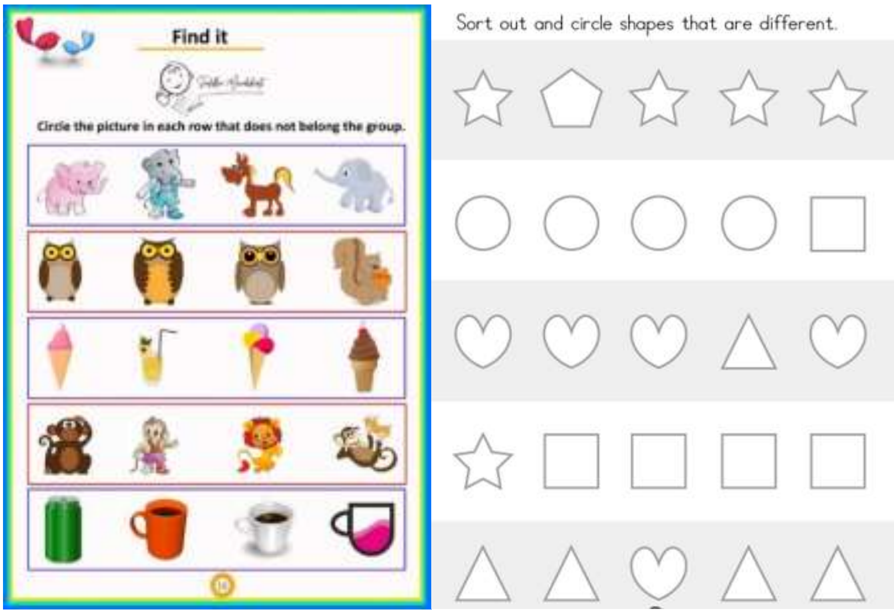
A nice activity of this type is the game „Fly, fly..." Someone says „Fly, fly...mouse". If the object does not fly, the children crouch, and if it flies, they stand on their feet. Or if the truth is told, the child stands, and if a lie is told, the child crouches.

On the Internet or in books with children's activities, you can find activity sheets of the type „circle all the animals in the picture" or „mark all the objects that can be found in the kindergarten".



(The School Mathematics Study Group (SMSG) material)[21]

Likewise, we have similar activity sheets for non-belonging. For example:

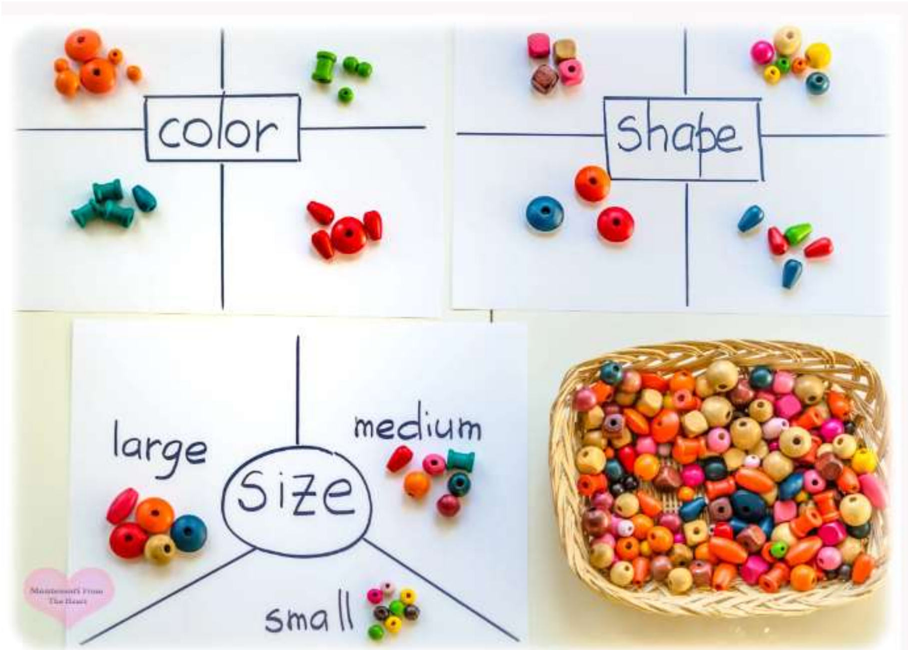


(taken from the internet, the site is unknown to me)

[21] SMSG was a group of mathematicians who tried to introduce modern mathematics into US education in the sixties and seventies of the last century. You can find a lot of their material on the Internet Archive website.

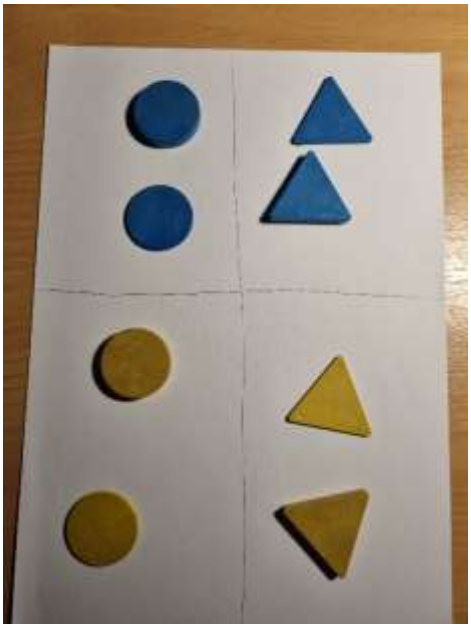
A2: Classification.

According to some selective properties, we classify objects into separate groups (sets). For example by color or shape, or sorting clothes by type into drawers (pants, T-shirts, socks, etc.)

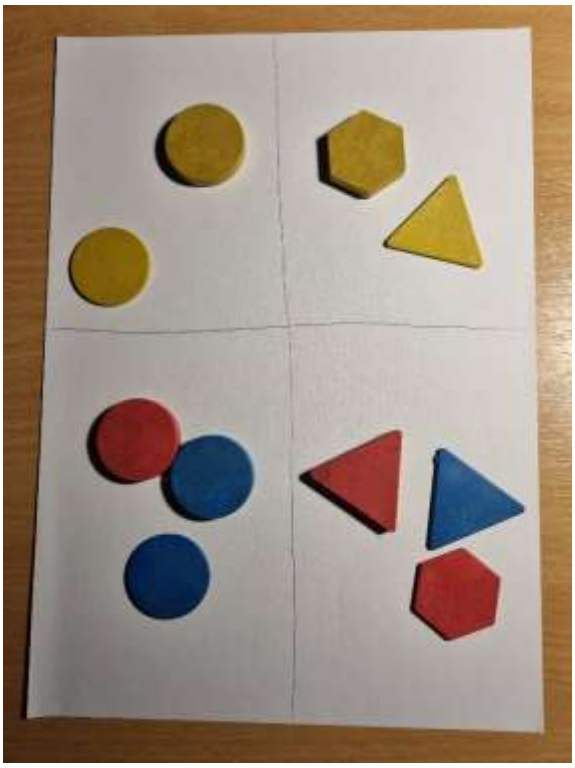


(https://montessorifromtheheart.com/)

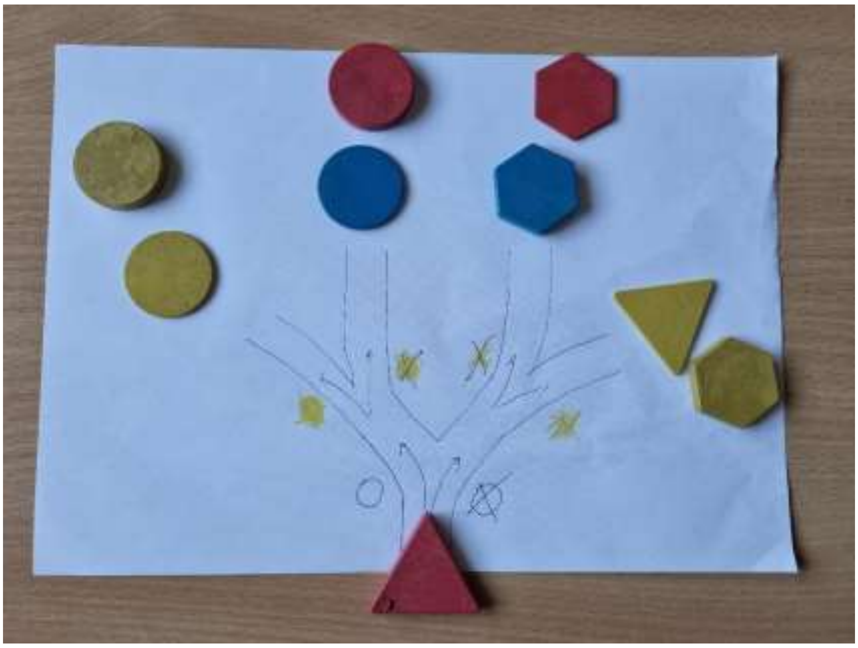
A more complex classification is when we classify by two properties, for example by color (blue or yellow) and by shape (circle or triangle). Thus we get 4 groups (blue circle, blue triangle, yellow circle, yellow triangle)



It is even more complex when we sort by two properties that an object may or may not have, for example yellow and circle. Then we have four groups again (yellow and circle, yellow and not a circle, not yellow and a circle, not yellow and not a circle:

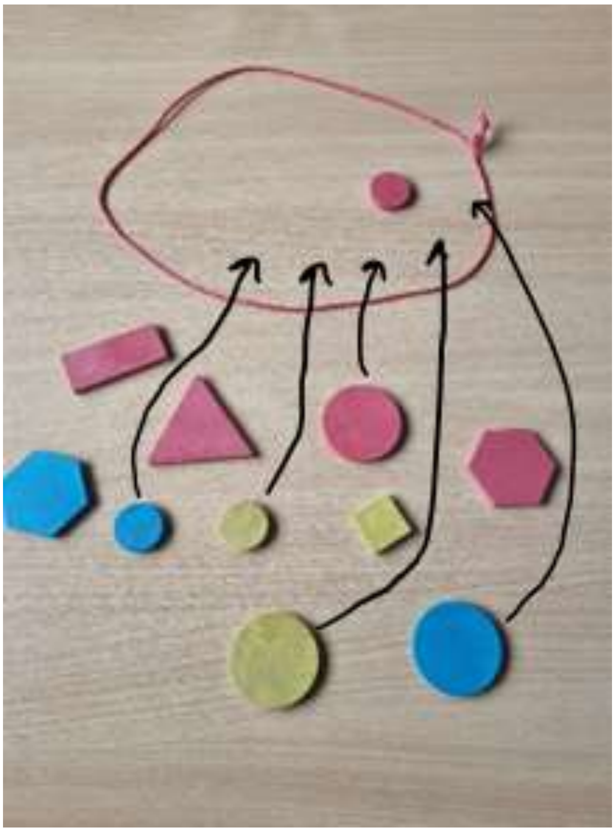


Even more complex sorting can be done using „rails“:

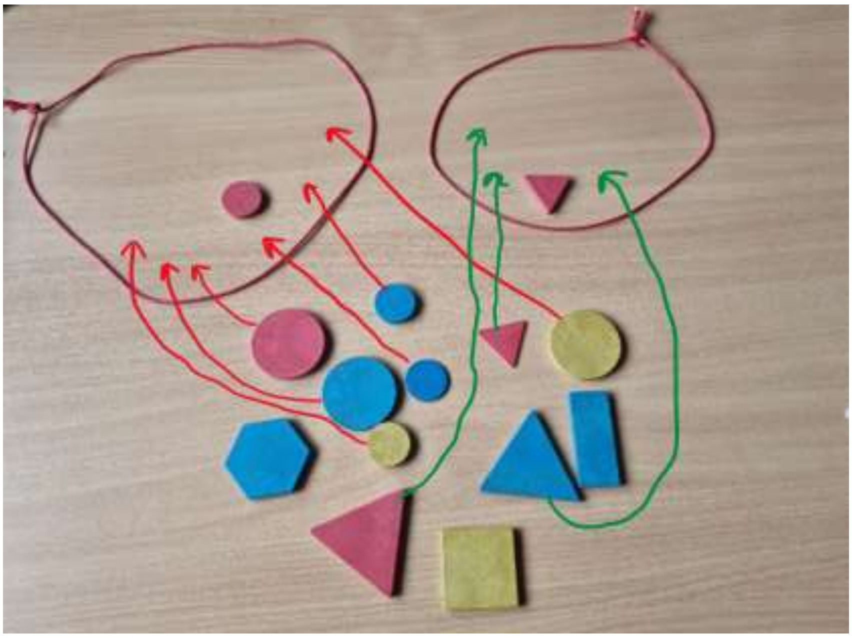


A3: Collecting objects into a set. At this level, it is first and foremost the spatial collecting of objects into boxes, piles, etc., either arbitrarily or by some property. This is how the child refines the understanding of various properties. Objects can also be united with a „lasso" - rope, arbitrarily or by some property. Or by drawing (closed line) on the picture. But it is always better to works with real objects than with drawings. Example:

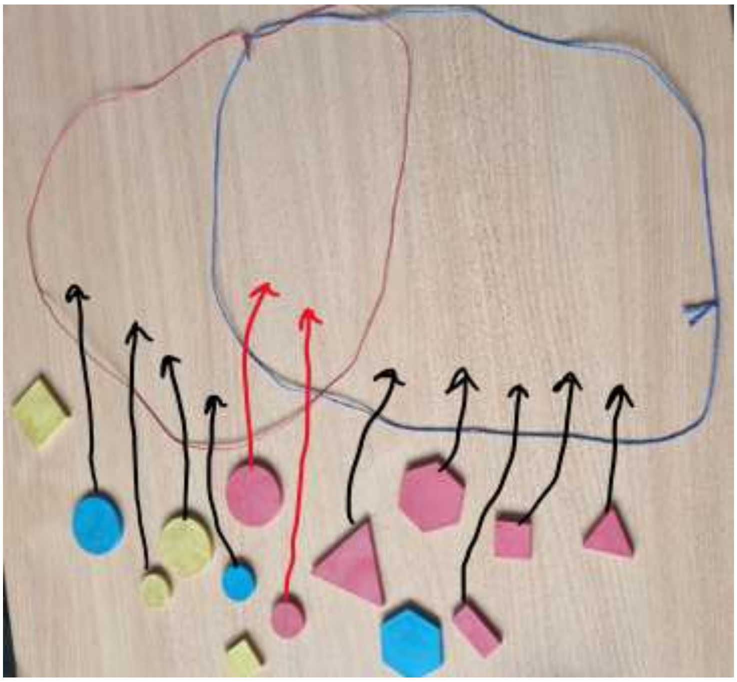
Put all the circles in the fence (the fence is a rope):



Put circles in one fence and triangles in another:

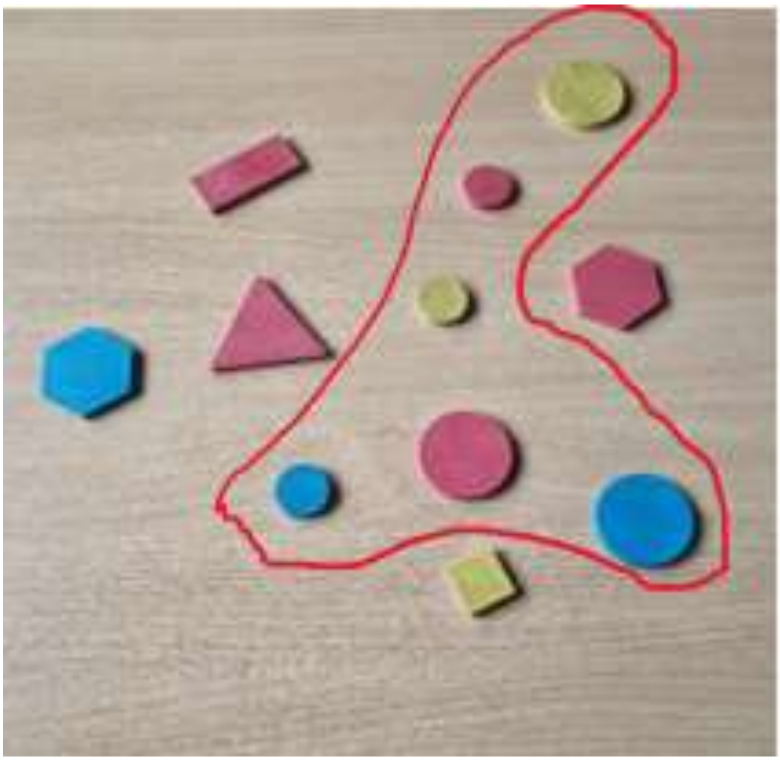


Put the circles in one fence and the red objects in the other (this is a more complicated situation because we have objects that should be in both fences, so we need to figure out how to place the fences):



Collection can be done by placing a fence (rope) around the objects, because this way we have a visual representation of the concept of collection. For example:

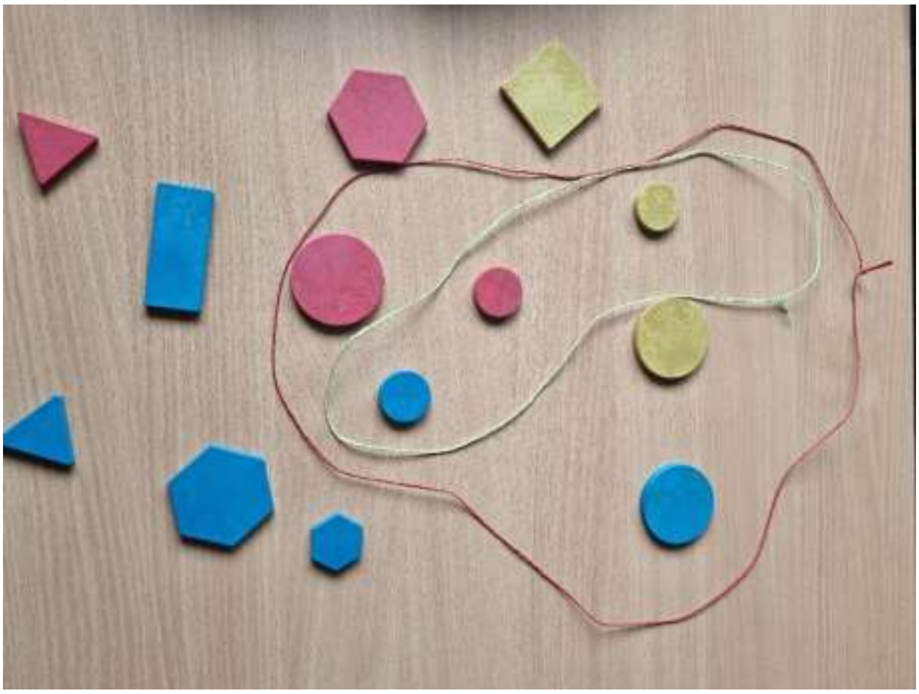
Fence all circles:



Or: fence all small figures, fence all blue figures, etc.

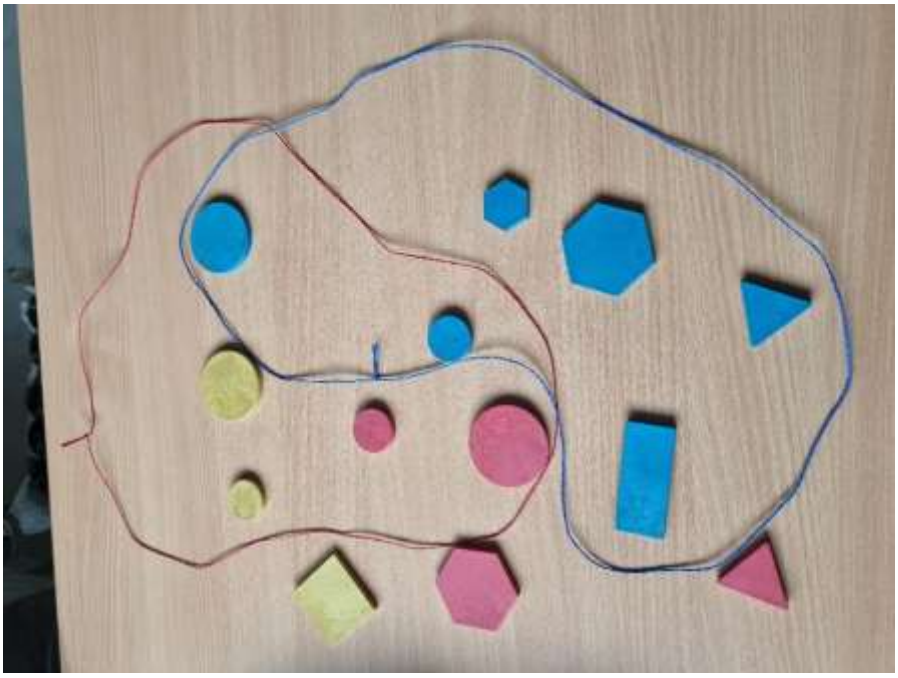
A4: Taking a subset from a set. This can be done with real objects or in a picture, arbitrarily or by some attribute. E.g.: Select all the cars (there are also other objects) and select all the blue ones among the cars.

Example. Fence all the circles, then separate all the small figures between them:

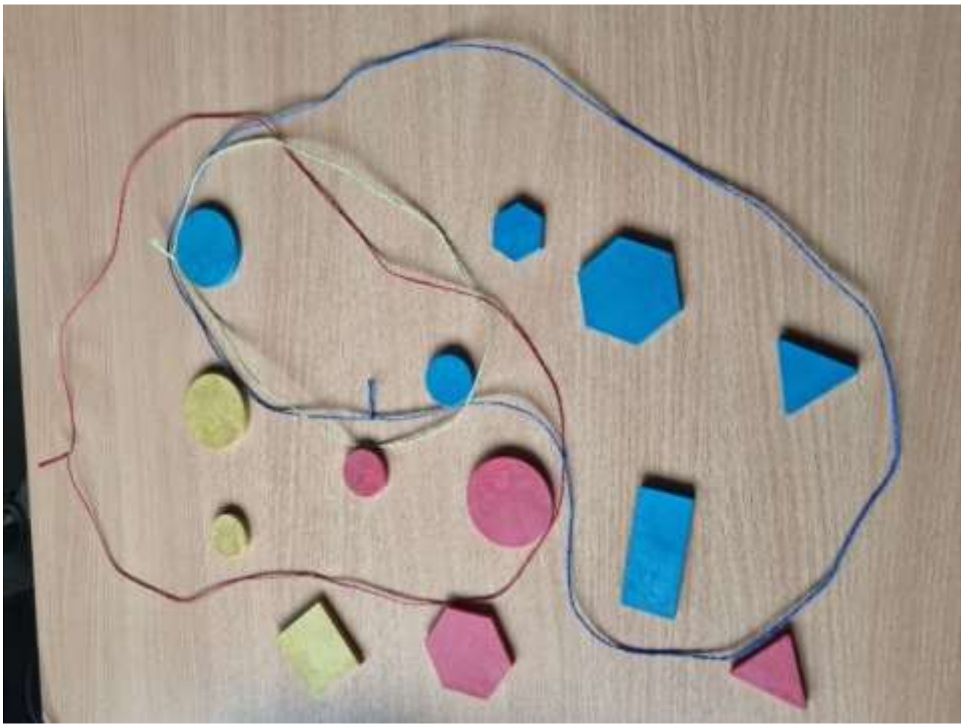


A5: Operations with sets. This can be done with objects or on a picture. We can have various piles or unify objects with a „lasso“ around them.

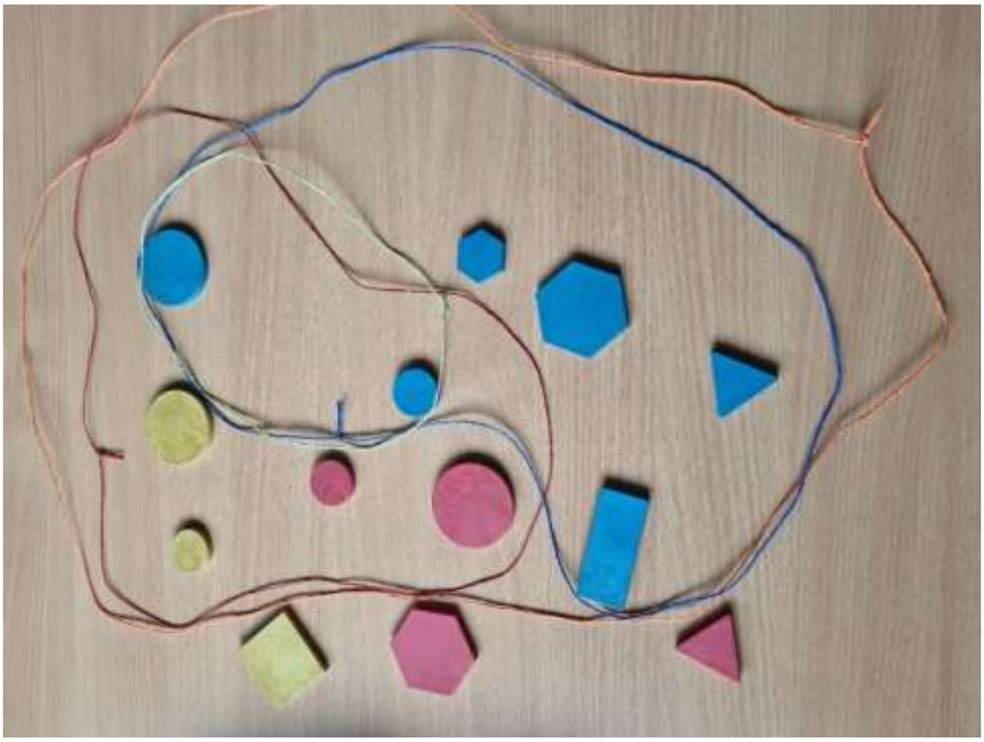
Example: We enclose all circles (one set – red fence) and we enclose all blue objects (second set – blue fence):



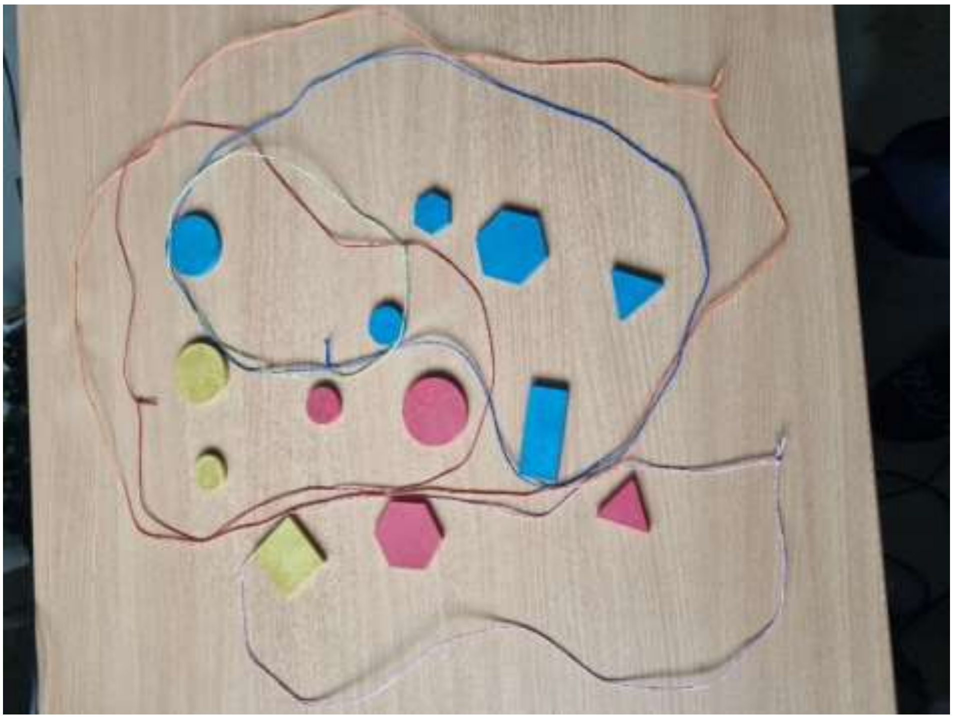
Now let's fence off all the blue circles (we take the common elements from both sets and get a new set - their intersection - yellow fence):



So let's fence off all those that are circles or blue figures (we take together all the elements from both sets and get a new set - their union - pink fence):



And we fence off all the remaining ones (we take the elements that are not in that set, which are neither circles nor blue, and we get a new set - the complement of the given set - a purple fence):



In order to motivate the child to collect by more complex properties, it is good, for example, to incorporate this activity into a story. One such story is sketched in Appendix 1

On website http://stern.buffalostate.edu/Movies/index.html you can find three links (Frederique Papy: *Teaching Strings in Kindergarten*, *Boys and Blue Strings*, *Venn Diagrams*) to videos of Fredereque teaching sets to young children.[22]

2 Relations

A short explanation of the term relation is on the page 10.

A1: Belonging to a relation. Using the examples of relations from the child's everyday life, because these relationships are close to children and they are happy to analyze them, we can lead the child to examine who is and who is not in a relation. For example: is Ana a friend of Nina (relation *x is a friend of y*), is Anica the mother of Ivica (relation *x is the mother of y*), is Lala Emina's doll (relation *x is a doll of y*), is the teddy bear in the living room (relation *x is in the room y*), is Stephan in front of Joseph (*x is in front of y*), etc. Whenever we

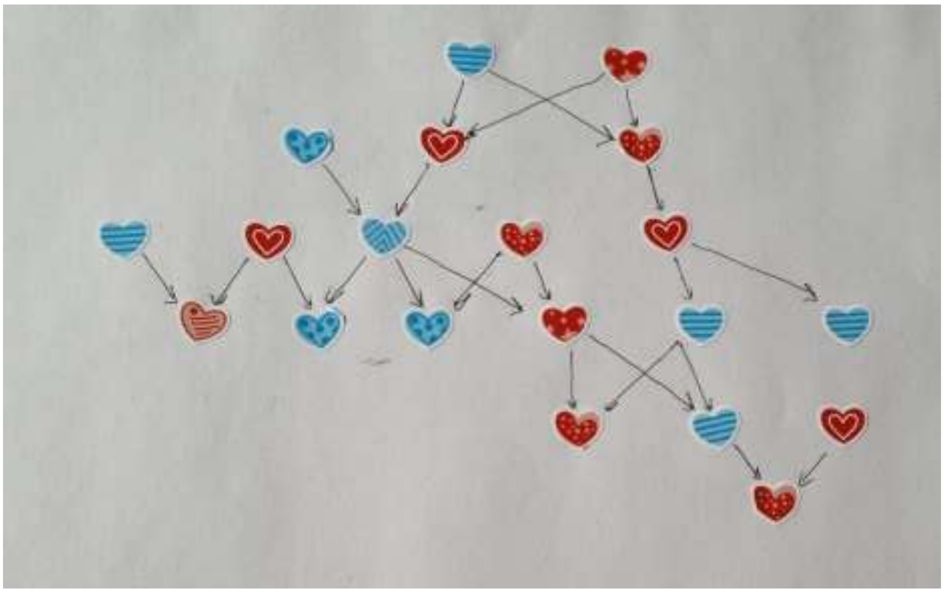
[22] Now those videos look a bit „scary“ to me, but that was the era - the seventies of the last century. Fredereque Papy (1921 – 2005) was known for successfully introducing modern mathematics to young children at that time. Her lectures are very thoughtful.

make a claim about two objects (people, dolls, place in space, ...) we say that these two objects are in some relation.

We can also develop a game here. Someone says a statement about two objects and the children answer with *true* or *false*.

We can also examine what happens when we change the places of the names in the relationship. If Juliet is the mother of Ingrid, can Ingrid be the mother of Juliet? Can Juliet be the mother of Juliet? If Matteo is the ancestor of Romeo and Romeo is the ancestor of Othello, whether is Matteo ancestor of Othello?

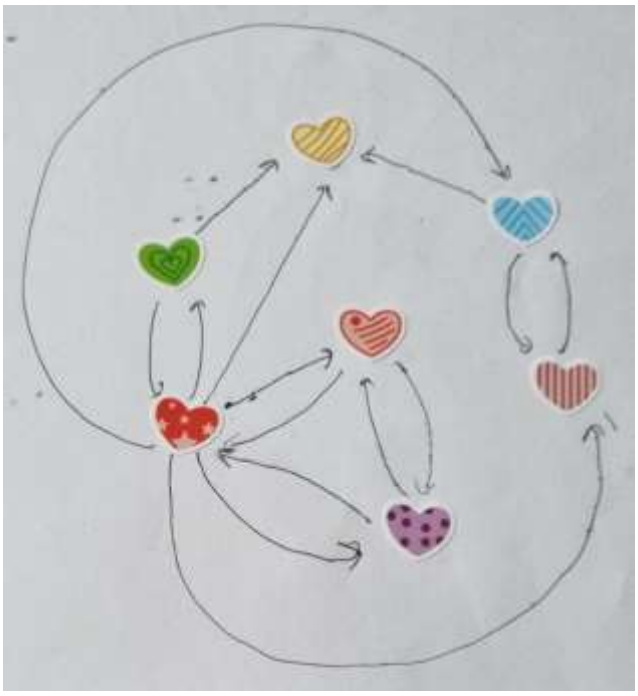
A2: Representation of a relation by a graph. A relation that connects objects from the same set can be visually represented by a graph. Let's present the objects with some kind of figures or with names, if the children can read. When one object is in relation to another, draw an arrow from the first object to the second. Take for example the relation *being a parent*. Let's present people on paper or a board or a wall panel with pictures or some figures. For family relationships, it is important that it is visible who is *male* and who is *female*. We can, for example, determine this with an appropriate color. The classic choice is blue for male and red for female. Let's draw an arrow from one person to another and thus represent that one person is the parent of another. Now we have the so-called relation graph (I only had heart-stickers available)



On a graph like this, children can probe anything, through a game or a story. They can find who is the mother and who is the grandfather, who is the grandmother of a given person, how many children does a person have, are two people brother and sister, etc. We can ask them if the ancestor of a person's ancestor is also their ancestor, etc.

We can choose one person on the graph and ask them to show their parents, grandparents, brothers, sisters, cousins, etc. (children can show who they are). We can ask who has the most children, who has the most sisters, etc.

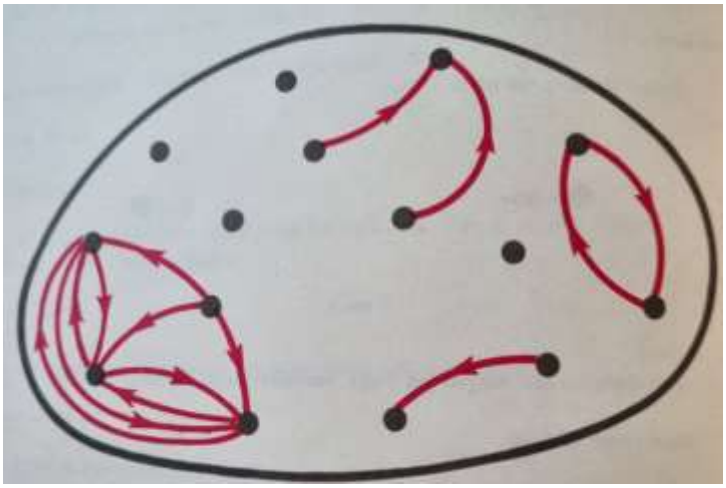
We can do the same with other relations over a set. For example: being a friend, being a neighbor,... Here is a graph that shows who likes to play with whom:



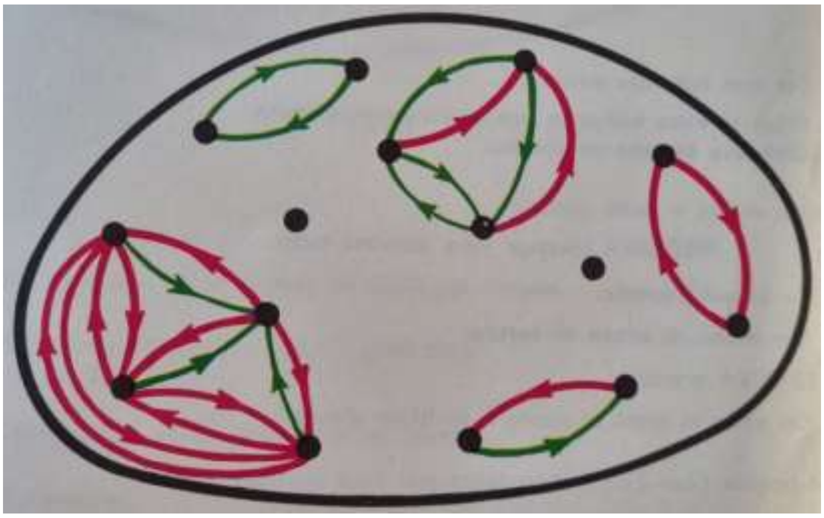
Now we can ask if there is someone who likes to play with everyone, someone who will not play with anyone, who they like to play with the most, is there a group where everyone likes to play with everyone, how many children a child likes to play with, etc.

Here is also a probing about who is who's brother or sister. This is taken (in abbreviated form) along with pictures from the book Frederik and Papy: *The Child and the Graphs*, Školska knjiga, Zagreb, 1972. On the website http://stern.buffalostate.edu/Movies/index.html you can find a link (*Arrows You are my sister*) to a video of Fredereque working on this topic with young children.

Let's represent each child as a point and draw a red arrow from one point *x* to another point *y*. With that arrow, child *x* points to his sister *y*. Let's say we got this graph:

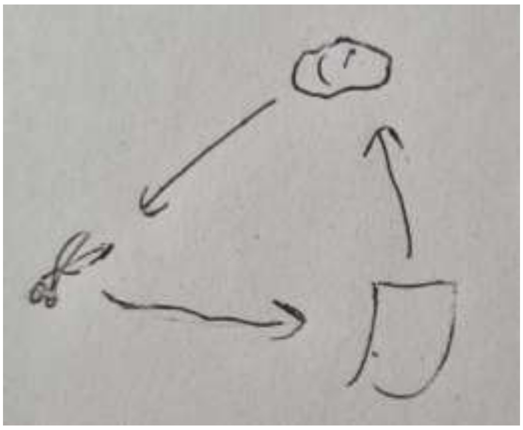


Now we can help children to extract a lot of information from the graph. It's like they're playing detectives. In addition to the standard questions (Who has the most sisters?, Who doesn't have a sister?), we can ask the children to find the girl (if an arrow goes to a child, it is a girl). A little more difficult, we can ask them to find a boy (if an arrow goes from a child and no arrow reaches him, then he is a brother, that is, a boy). Or to find two brothers (these are the points in the upper part of the drawing from which the arrows start), etc. We can also give the children the task of drawing green arrows with which the children will show their brothers:

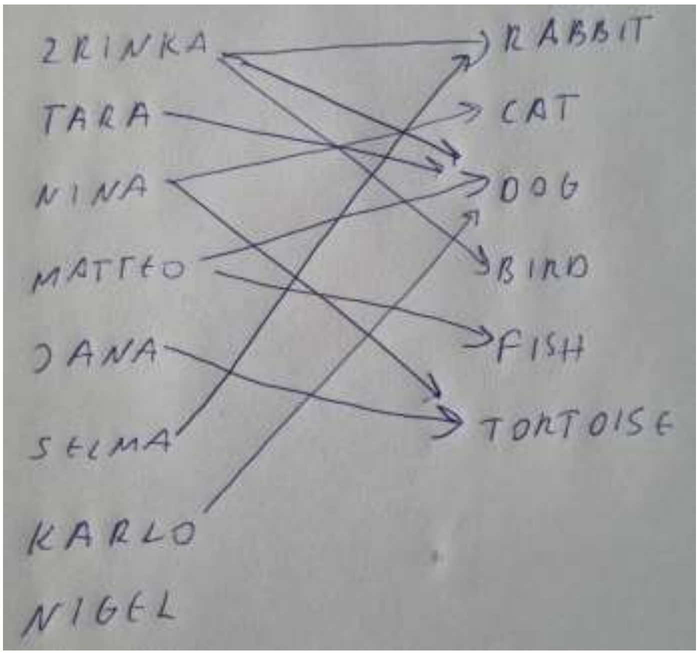


A3. Examining the properties of relations. In assigning relations with words or arrows, we can go a step further: instruct children to examine the properties of relations, as already indicated in **A1**. For example if *x* is related to *y*, then is *y* also related to *x* (symmetry of the relation)? Can we exchange the places of objects in a relation? This should, for example, be valid for *friendship*, and for *sex*, but not for *motherhood*. Or, if *x* is related to *y* and *y* is related to *z*, then is *x* related to *z* (transitivity of the relation)? This applies, for example, to the relation of *being an ancestor* and *being higher*, but not for *being a friend*. Or, is *x* in a relation with *x* (reflexivity of the relation). For example, the similarity of triangles is such a relation (every triangle is similar to itself) but not the relation *to be higher*. This play with moving words in a sentence, in addition to being mathematically and linguistically instructive, can be interesting for a child. For example, if we replace places of nouns in the sentence *Nina loves ice cream*, we will get the funny sentence *Ice cream loves Nina*. Similarly, from the true sentence *Head is on shoulders*, we will get false sentences *Shoulders are on head* (relation is not symmetrical) and *Head is on head* (relation is not reflexive). If we represent relations with graphs, all this can be shown by the visual relationship of arrows. For example, symmetry means that each arrow has a „return“ arrow, and reflexivity means that from each object one arrow goes towards it.

A4: Rock – paper – scissors game. Here we have the relation *to be stronger*. It is convenient to represent it with a graph. The child draws the arrow from stronger to weaker. The graph clearly shows that there is no strongest object as in standard comparison relations:

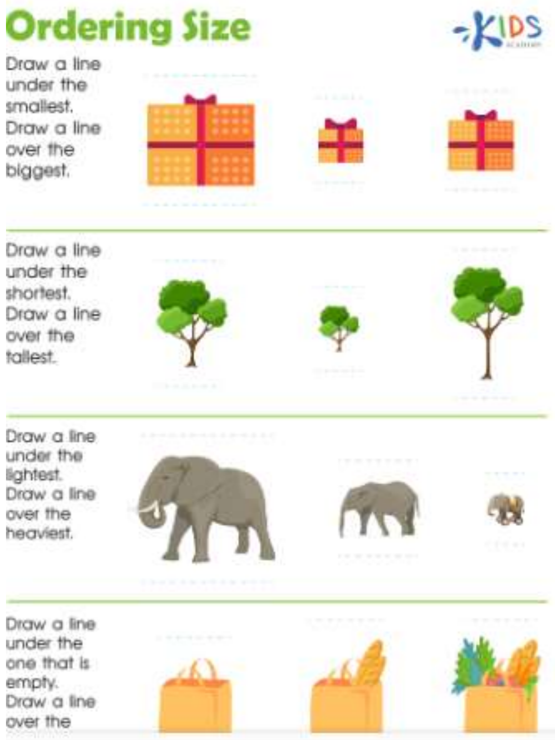


A5: Relations between objects from two sets. When objects from one set are connected by a binary relation to objects from another set, it is convenient to represent the relation with arrows that go from one set to another set. For example, the relation *child x has a pet of type y* is a relation between children and the type of their pet. We can present it visually as follows (pictures would be better instead of names):



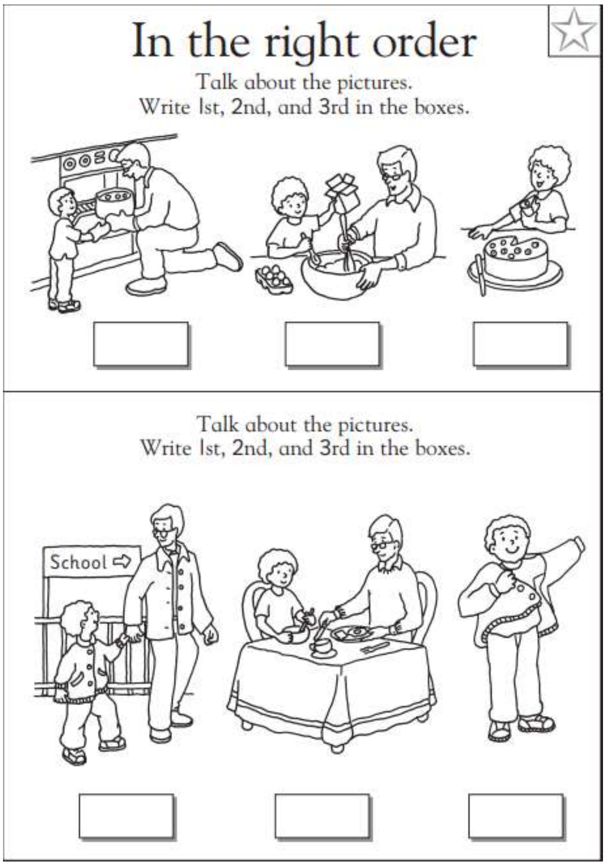
Questions: What kind of pet does Matteo have?, Who has birds?, Who doesn't have a pet?, Who has the most types of pets?, What type do children have the most?, etc.

A6: Ordering relations. Such relations introduce some order among the objects. We say that an ordering relation is linear or total ordering when we can use it to arrange objects in a sequence. *Being taller, being longer, being heavier* are such relations. Children often arrange objects according to a certain size and thus acquire the idea of a linear ordering.



(taken from the internet, the site is unknown to me)

The relation between events - which event happened first - is an example of a linear ordering:



(taken from the internet, the site is unknown to me)

An ordering relation does not have to be linear. For example, that one object is above another, that is, the second is below the first, is not a linear ordering. We can find two objects such that neither of them is higher than the other.

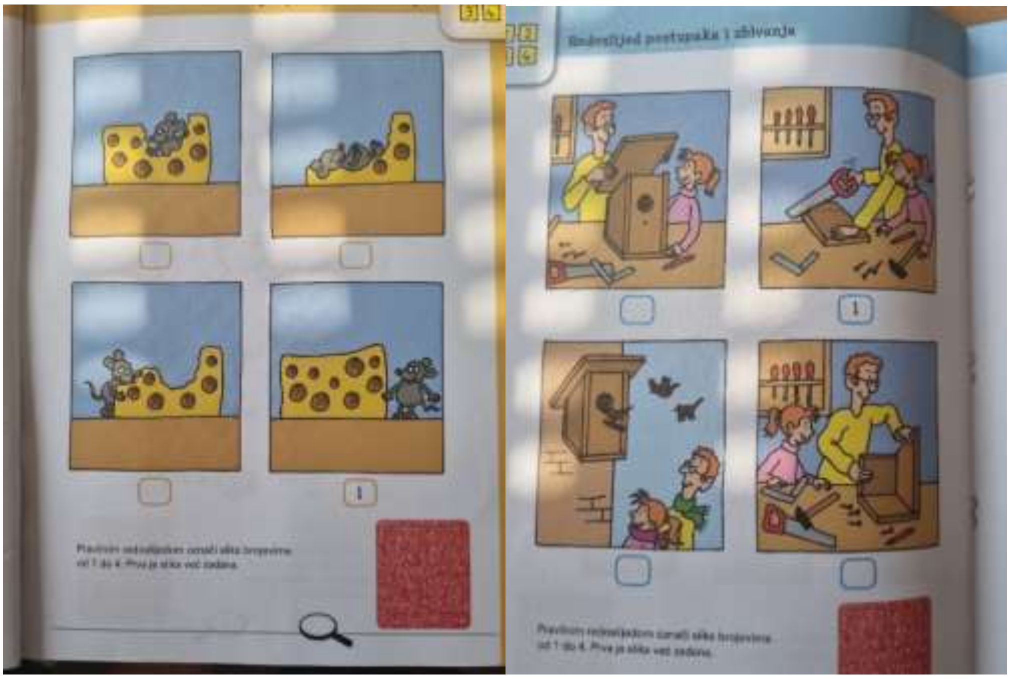
(taken from the internet, the site is unknown to me)

Here, in addition to questions like „is *x* below *y*", we can also ask if there is an object that is below all, or above all, how many objects are above an object, which objects are below *x* and below *y*, etc.

For such relations that mutually reverse the order of objects - *x* is below *y* just when *y* is above *x* - we say that they are mutually inverse relations.

Likewise, the order of dressing is not a linear ordering: we should put on socks before shoes, but we don't have to put on shoes before jackets.

A7: Time order.



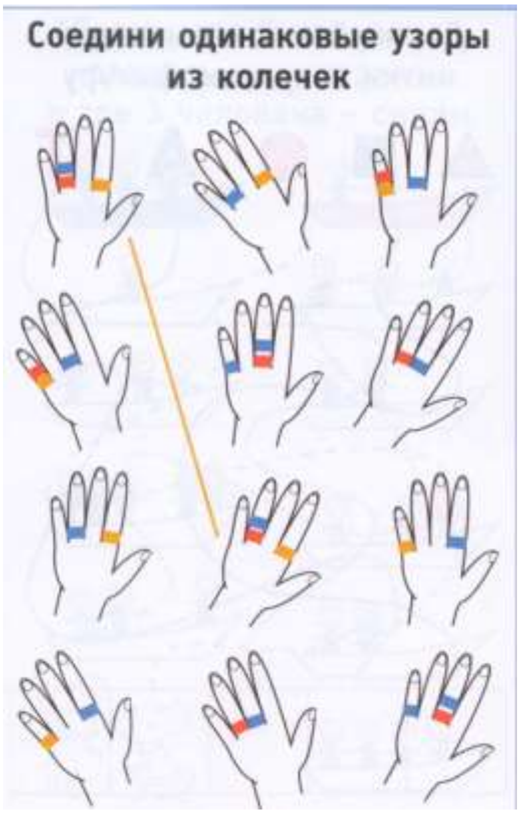
(*Workbook from 4 to 5 years*, Schwager und Steinlein Verlag, from Lidl)

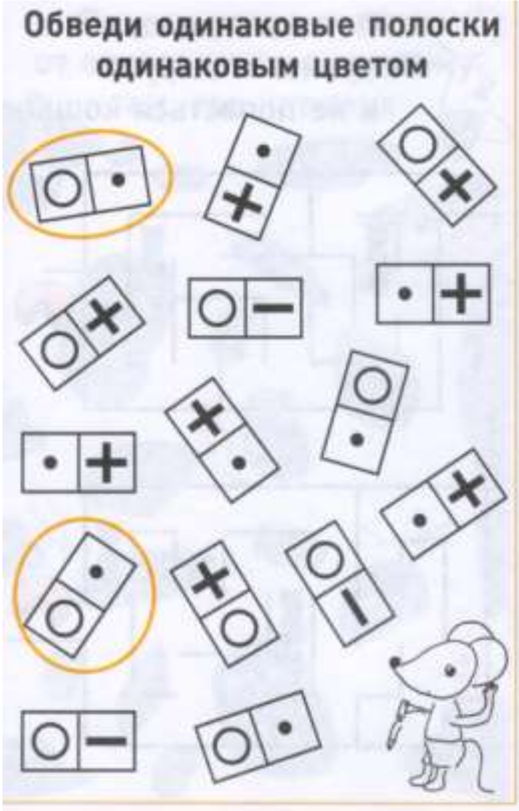
A8: Equivalence relations (similarity relations). Intuitively, these are relations that classify (sort) objects into groups such that in the same group there are objects that are somewhat similar. For example *x is as tall as y* classifies objects by height. We put all objects of the same height in the same group. On the contrary, whenever we classify objects into separate groups, behind it lies the equivalence relation *x and y are in the same group*. For example, children in kindergarten are divided into kindergarten groups. Behind that lies the equivalence relation *x and y are in the same kindergarten group*. Relation *x and y are animals of the same type* sorts animals into cats, dogs, etc. Relation *x and y are figures of the same shape* sorts characters into triangles, quadrilaterals, stars, circles,...

Children naturally classify objects according to some relation of similarity between objects and thus acquire the idea of equivalence relation.

Example: sort clothes in drawers. Where will you put pajamas, where pants, where T-shirts? In the background of this classification lies the similarity relation *x and y are clothes of the same type*.

(taken from the internet, the site is unknown to me)





(MouseMatics)[23]

A9: Relations connected to counting and measuring. Counting the amount of objects, as well as measuring length, weight, volume, etc. in its finished form gives a number. However, there are also two qualitative relations that young children can easily acquire and thus measure in a qualitative way. These are the ordering relations *x is bigger* (*longer, heavier, …*) *than y* and the similarity (equivalence) relations *x is as big* (*long, heavy, …*) *as y*.

The activity related to comparing sets is given below in the *Functions* section, page 11.

Comparison of lengths can be done either directly or indirectly via a third length. For example, for children to see which T-rex is taller, they can put them next to each other (direct comparison). But in order for the children to see whose storage cabinet is taller, they can take a rope (it can also be a construction or tailor's tape measure - children like to use adults' things), mark the height of one cabinet on it with their fingers, transfer the string to the other cabinet and compare its length with the marked length on the rope (indirect comparison).

[23] There are a lot of pictures in this material that you will easily recognize by the design, the small muscle and the Russian language. They are taken from excellent books by author Jane Katz (Russian: Женя Кац). You can get the books, for example, via Amazon.de (*MouseMatics: Unusual Math For 4 Year Olds, MouseMatics: Learning Math the Fun Way. Workbook of Logic Problems for children ages 5-6, MouseMatics: Unusual Math For 7-8 year olds*). Only the middle book is in English, the others are in Russian. But you don't need to know Russian to use them. Almost everywhere, what needs to be done is shown on a visual example.

Cuisenaire rods are very good for comparing lengths (and later for learning arithmetic operations). Children can make trains or towers out of them and ask which one is longer (higher) and how much to subtract or add to make them the same. Teacher Simon Gregg blogs about lots of activities with Cuisenaire rods (https://followinglearning.blogspot.com/search?q=Cuisenaire)

Children can compare the weight of an object using a classic scale with arms, or indirectly through a scale with weights, or using a measuring spring.

They can compare the volumes of containers by pouring water into the containers. They can compare the volumes of various piles of sand, by pouring them into the same containers. They can compare the volumes of objects by placing them in the same containers with the same amount of water and seeing how far the water level in each container has risen, etc.

Children naturally like these activities, so it is not difficult to motivate them.

This type of comparison also includes the problems of whether something will fit into something, which children can guess, and then check experimentally. This includes the story of the golden-haired girl, where one pot is too hot, another is too cold, and the third is just right, while one bed is too narrow, another is too wide, and the third is just right.

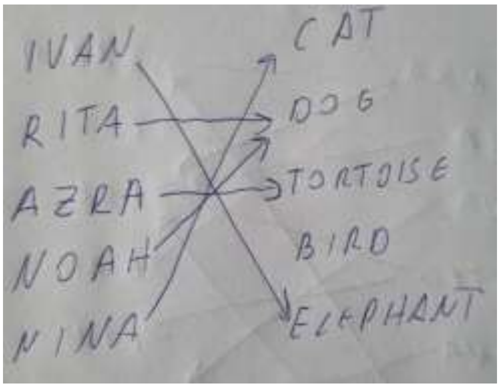
3 Functions

As with sets and relations, activities with functions from their everyday life are most natural to children. To recall (see page 11) functions (assignments, operations, mappings, transformations) applied to one object (or more objects) uniquely yield another object. For example, we can add the mother to each child, not the ear (because it is not unambiguous), but we can add, for example, the left ear. Whenever we identify one object using others, we have determined a function. For example, kindergarten a child goes to is a function that associates each child (who goes to kindergarten) with the kindergarten she goes to. This is how children build functions when, according to some rule, they associate one object with another, or they invent what they will associate with what, for example, a certain gift for each child from a group. Of course, it is good to create a certain motivational context for all this.

A1: Functions given by language description. Whenever we identify an object using others, we have determined a function. We usually do this with some linguistic expression: *mother of x*, *left ear of x*, *favorite toy of x*, *youngest brother of x*, *oldest person in a group x*, *favorite cartoon character of a child x*, etc. For example: *the heaviest object in a room x* is a description of a function: we assign an object to each room in the house (apartment). We can set activities for finding such an object for a particular room (individual *x*), i.e. apply this function to individual objects: „find the heaviest object in the bathroom“, „find the heaviest object in the pantry“, etc. We can do similar activities with other language descriptions: instead of „*x*“ we put the name of an object (input to the function) and the resulting description identifies the output of the function for that input. On the contrary, each concrete description of one object by means of another (others) „hides“ a function within itself. For example, the description *Ana's best friend* „hides“ the function *the best friend of child x*, while three plus five hides the addition function *x*+*y* which assigns to two numbers their sum.

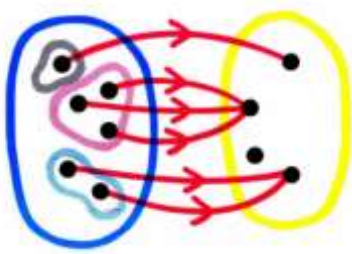
Game: show left ear, right ear, left neighbor, right neighbor, etc.

A2: Functions given by arrows. Functions can be graphically represented by arrows that show the assignments of objects. For example, we can make a picture in which there are children on the left and animals on the right (they are represented with words on the picture below, although it would be better if they were represented by pictures). Arrows show the function that assigns to each child the animal she likes the most.



Now we can ask ourselves: Which animal do children love the most? Is there any animal that no child loves the most? How many children like the cat the most, etc.

Here is another example. Each bird flew to its nest:

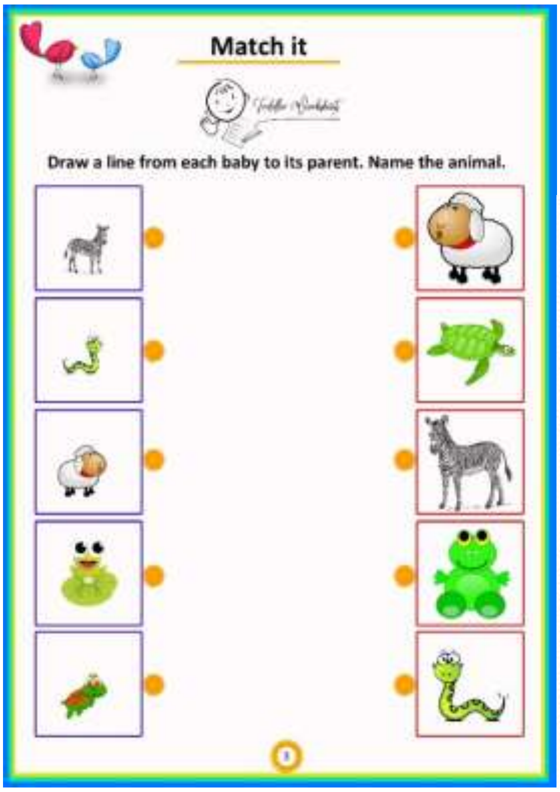


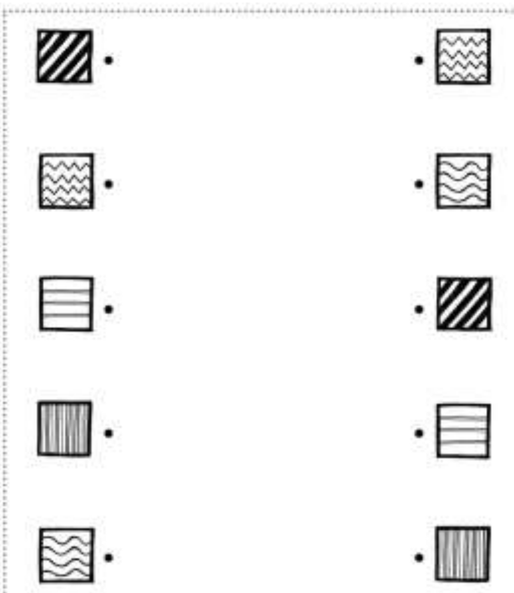
(Fredereque: The Child and the Graphs)

Which nest has the most birds? Is there a nest without a bird?

Here are some more examples:

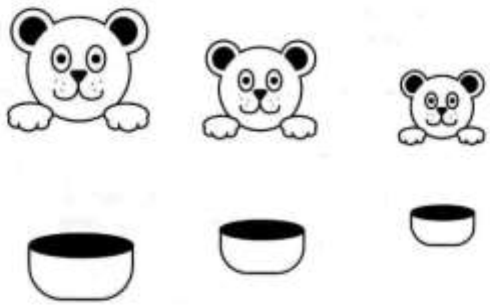
(Oštroumke, Element, Zagreb, 2021.)





(taken from the internet, the site is unknown to me)

Assign to each puppy its food bowl:



(taken from the internet, the site is unknown to me)

(Oštroumke, Element, Zagreb, 2021)

(Oštroumke, Element, Zagreb, 2021.)

(Oštroumke, Element, 2021.)

A3: Functions given by actions. The simplest activities with functions are in performing certain actions that transform objects or states. For example from paper an airplane (this function attaches to each paper an airplane made from it), from undressed doll a dressed up doll (this function attaches a final dressed up look to each undressed doll), from initial position final body position (this function attaches final body position to initial position).

This „action“ view of functions can be picturesquely described by machines that perform certain tasks. Such a machine rotates something, duplicates, grows up, etc. For example, the following machine turns every adult into a baby:

(Mitsumasa Anno: Anno's math games II)[24]

[24] You can borrow the book to read at Internet Archive.org. Mitsumasa Anno (1926 – 2020) wrote many children's books with beautiful warm drawings, some of which also have mathematical content.

What does this machine do?

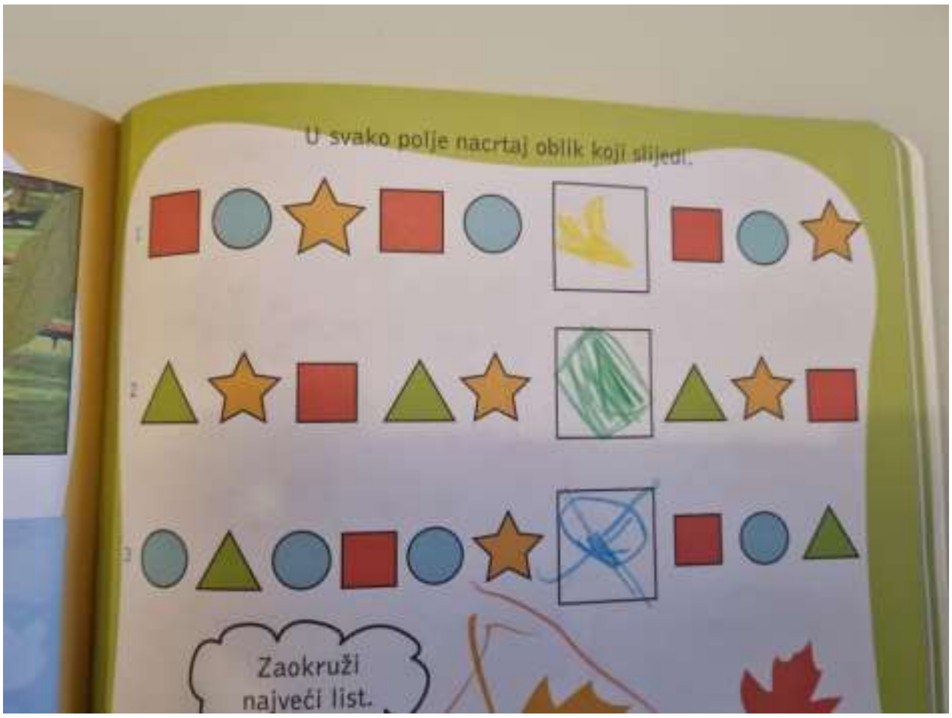
(Mitsumasa Anno: Anno's math games II)

We can make a game. We have a couple of boxes (cardboard houses). Each box represents one function. In each box there is one child who performs this function. When a paper with a figure is handed to the box, the child in the box applies a certain action to that figure: paint the figure yellow, cut it in half, make another such figure, cross it, make such a figure twice as big, etc. Children should go from box to box and determine what action each box does. A step further: a figure is sent through several boxes, at the end the children get the figure and must determine which boxes the figvure passed through.

A4: Functions given by commands. For example, one child pretends to be a robot and the other gives her commands of the following type: go, stop, turn right (90 degrees), turn left (90 degrees). Using these commands, the child must bring the robot from one place to another (in the room, on the playground, etc). Here we already have a situation of composing functions. A step further is issuing more complex commands (repeating until something is done, making a choice) but this is described in the section on procedural thinking.

A5: Sequences. Several children line up. Each sequence determines one function. A position in the sequence is associated with the child at that position. We can ask who is third in a row, who is after someone, who is before someone, who is last in a row, who is in the middle. We can examine all possible sequences of, for example, three children (we can make 6 different sequences)

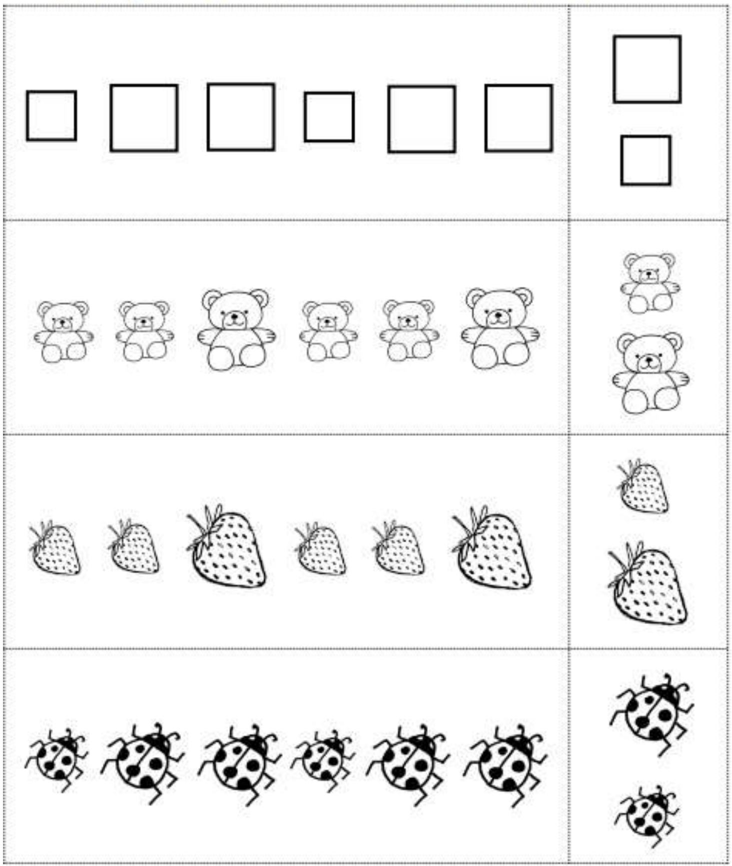
A6: Patterns. What we have put in the sequence before determines what we put next (we say that the sequence is given recursively). You can find many such activities on the Internet and in workbooks for children:



(Oštroumke, Element, Zagreb, 2021.)

AAB, ABB Patterns: Size

Look at the patterns. Color to show which one comes next.



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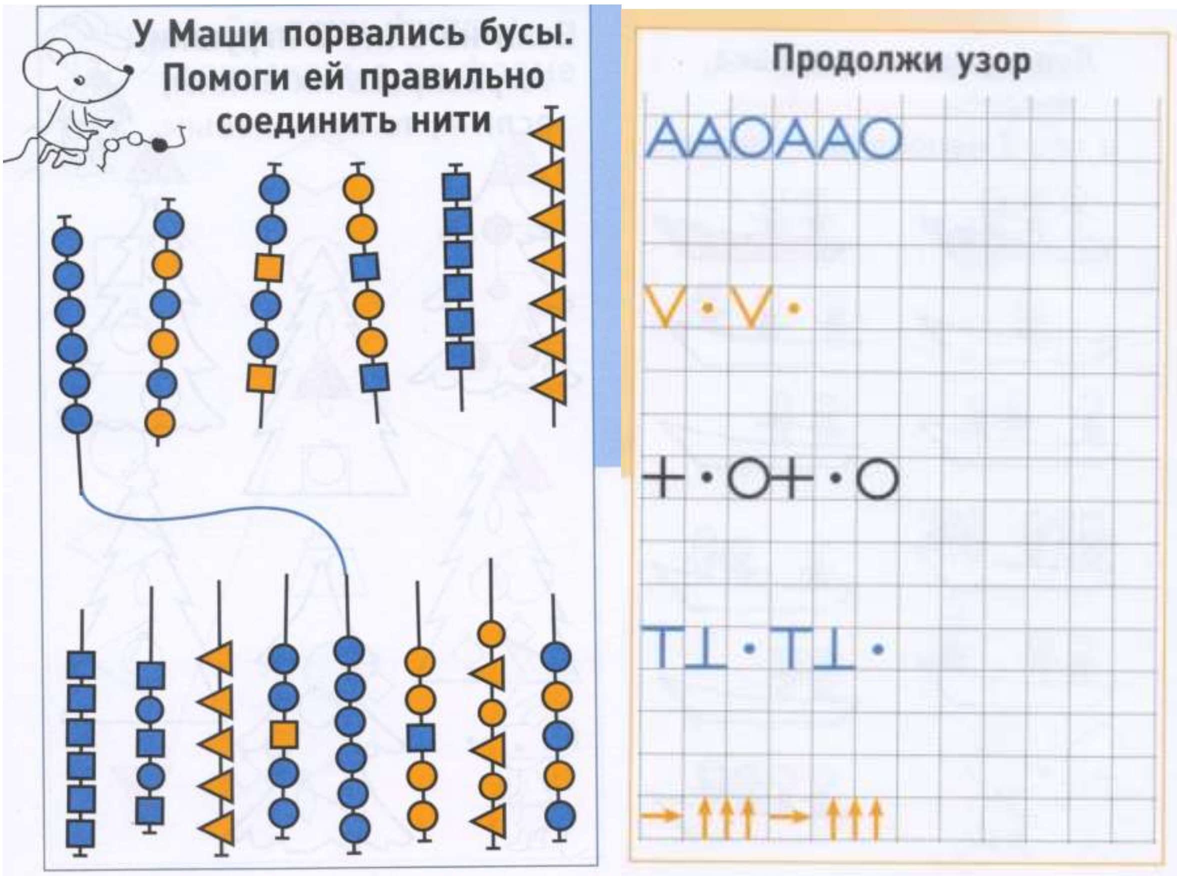
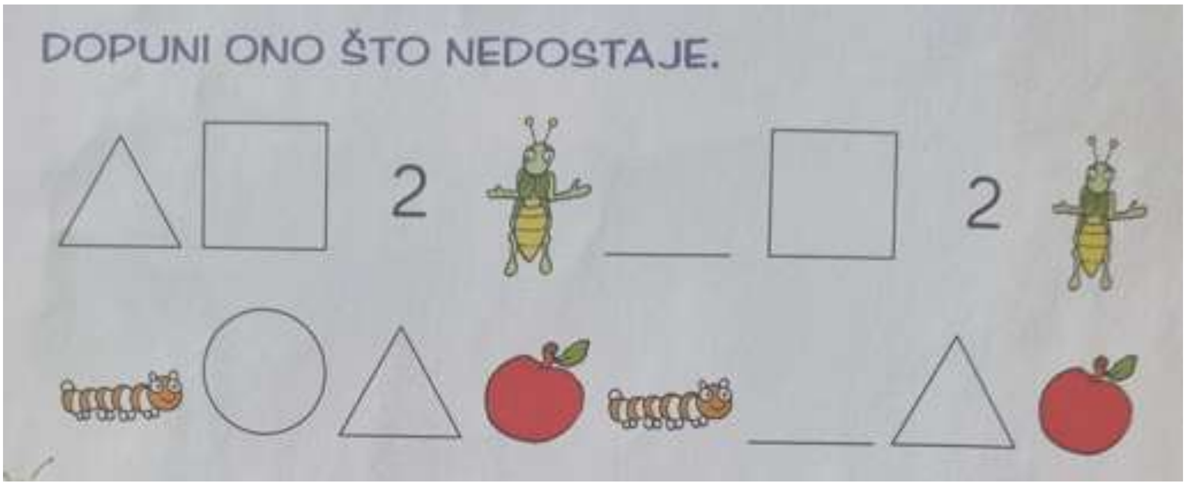


(MouseMatics)



(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)

(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)

A7: Encription and decription. This activity assumes knowledge of letters. The easiest way is to make a table of substitution of letters in the text. For example:

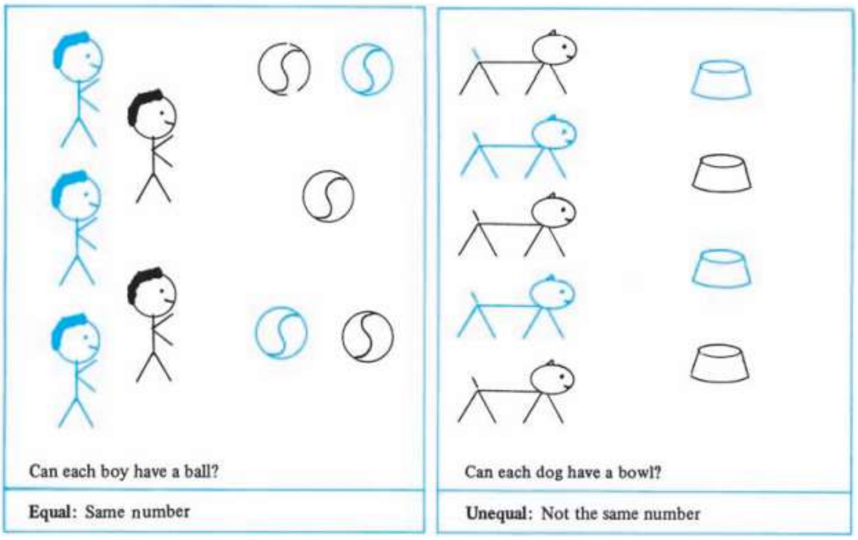
A	B	C	D	E	F	G	…
P	H	A	F	M	J	Z	…

This table defines a function that converts letters to letters. In order to decipher the text, this function must map different letters to different letters. Thus, it has an inverse decryption function that transfers the lower row of letters to the upper row. For example, by encrypting the word ABAC we will get the word PHPA and by decrypting the word ABAC again.

A8: Comparing multitudes. Using functions, we compare sets, so that we connect elements of one set with elements of another set in such a way that various elements of one set are associated with various elements of another set (so-called 1-1 assignment). If, with such a 1-1 connection, all the elements of one set are connected and there are some left in the other, then that second set is larger. If all elements from both sets are connected in this way, then we say that the sets have the same elements.

Example: connecting children and swings in the park, children and play balls, etc. Will each child have his own swing (ball)?

(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)



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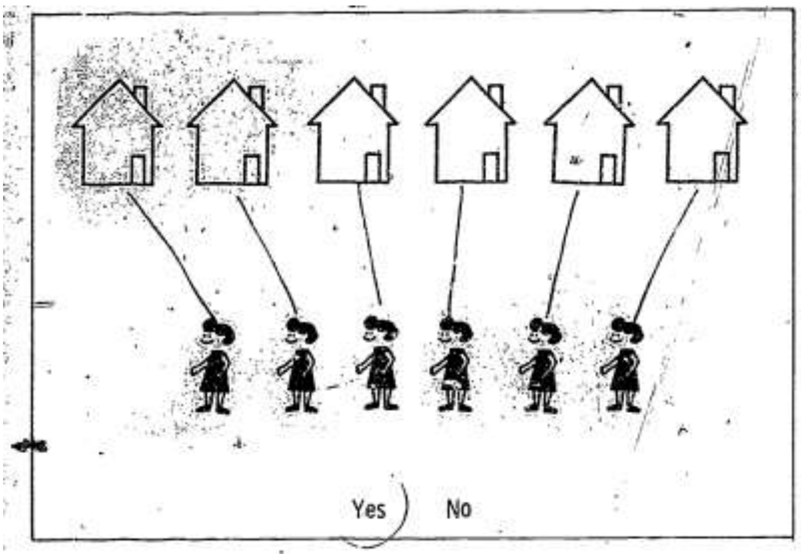
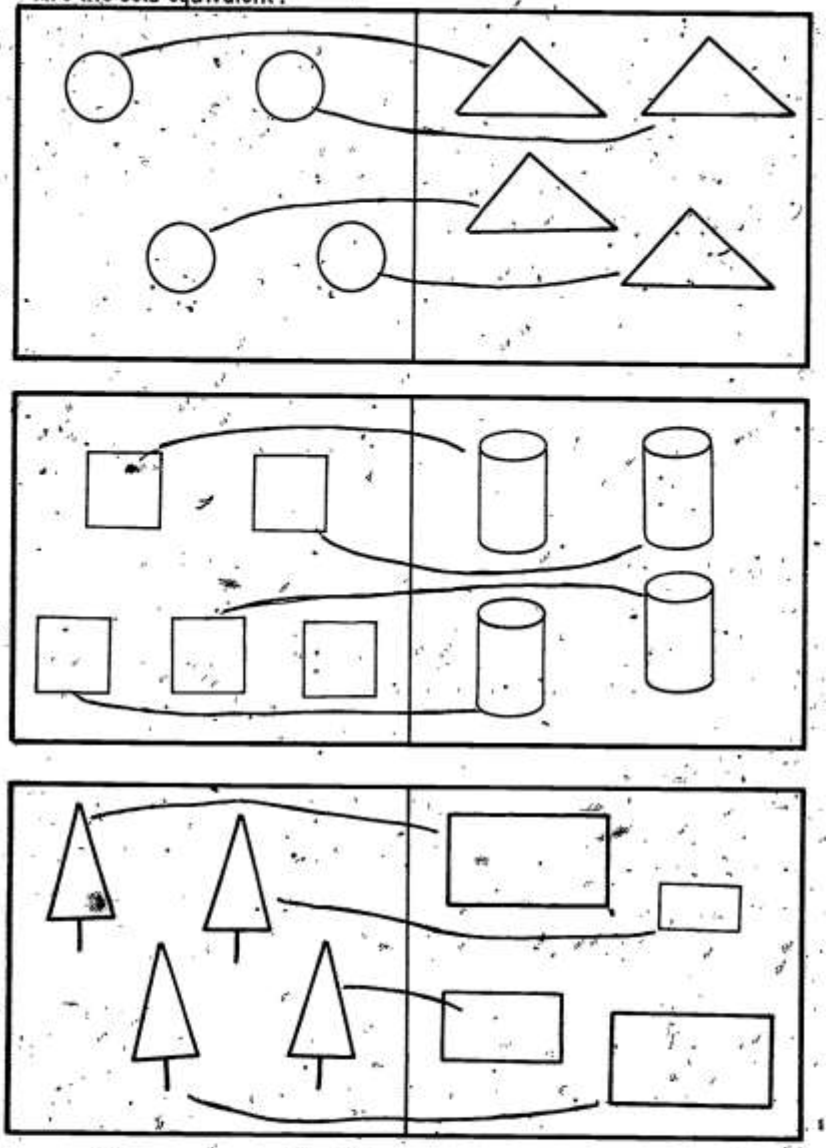


Are the sets equivalent?



(School Mathematics Study Group materijal)

A9: Counting and measuring. Counting is the formation of a corresponding function between the objects to be counted and numbers, but these activities will be described in the section *Numbers*. Measurement is the formation of a corresponding function - for example, by measuring we assign to each person their height - but these activities will also be described later, in the measurement activities, in the *Numbers* section.

A10: Composing functions. We can compose functions: *show dad from mom*, *mom from mom*, *youngest brother from mom*. Or, *go two steps forward, then three steps to the left*, etc.

A11: Properties of functions. Does the function start from every object from a set (*totality*), does it reach every object from a set (*surjectivity*), does it map different objects into different ones (*injectivity*), or can you find different objects that it maps into the same one (it is not injective)? In all previous examples we can include these questions. For example, the function *mom of x* is a total function on the set of people (everyone has a mom), it is not a surjection (not all are moms) and it is not an injection because brother and sister have the same mom.

A12. Inverse function. When we return the output of a function to the input of the function, do we also have a function (the so-called *inverse* function)? We have seen that it is necessary for encryption. A necessary and sufficient condition for the existence of an inverse function is that the function is an injection. This is easy to see with the arrow representation of the function: we must not have two arrows with the same end. As in the previous example, in which the brother and sister point to the same person, their mother. When we reverse the arrows, we lose the uniqueness: *the child of x* is not a function because a person can have more children. We can get the function by additionally specifying the return: for example, *the youngest child of person x*. This is the so-called *partial inverse* of the initial function *mom of x.*

If the functions have an inverse, then we can play with finding rules for canceling actions. For example the inverse of *go three steps forward* is *go three steps back*. The inverse of composed actions (functions) reverses the order of their cancellation. For example, the inverse action of *go two steps forward, then three steps left* is *go three steps right, then two steps back*.

4 Structures

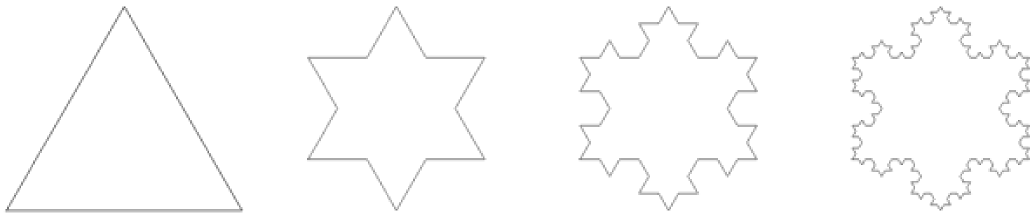
As described on page 12, we build structures from sets, relations and functions. We take a couple of sets, relations and functions between them and they, viewed together, form one structure. In the previous sections, we constantly looked at relations and functions over some sets, so we already had simple structures in the background of the activities.

A1 Grafovi. Graphs, which are described in the Relation section, are examples of structures composed of objects and arrows connecting these objects, so you can look at the corresponding activities there.

A2 Rekurzivne strukture. Building various structures using Lego bricks is an example of the so-called recursive structures: we have initial elements and building rules, with the help of which we build various constructions from initial elements.

Line patterns are examples of recursive structures. For example, we draw a flower (initial object) and whenever we have drawn a series of objects we look at the last object: we draw the sun, if the last object is a flower, or a flower if the last object is the sun. Thus we generate a sequence: flower, sun, flower, sun, ... Of course we can have a different start or a different rule. For example, we can modify the previous rule so that we should always draw one more flower than last time. We will get a sequence: flower, sun, flower, flower, sun, flower, flower, flower, sun, ... Building patterns according to a rule, or discovering the rules from an already built pattern, is an interesting activity for children.

We can generate fractals in the same way. Here's a Koch flake. The beginning is an equilateral triangle, and in each step the side of the resulting triangle is replaced as shown in the picture. Children can do this freehand - it doesn't have to be precise:



Thus, children are introduced to very interesting figures - fractals - whose parts are somehow similar to the whole.

A3 igre. Children's natural structures are games. There we have a set of some objects with rules for what we can do with them. Personally, I prefer cooperative games to competitive games, both because children do not like to lose in that period, and because it is better for them (and for the future of our civilization) to learn to cooperate than to compete.

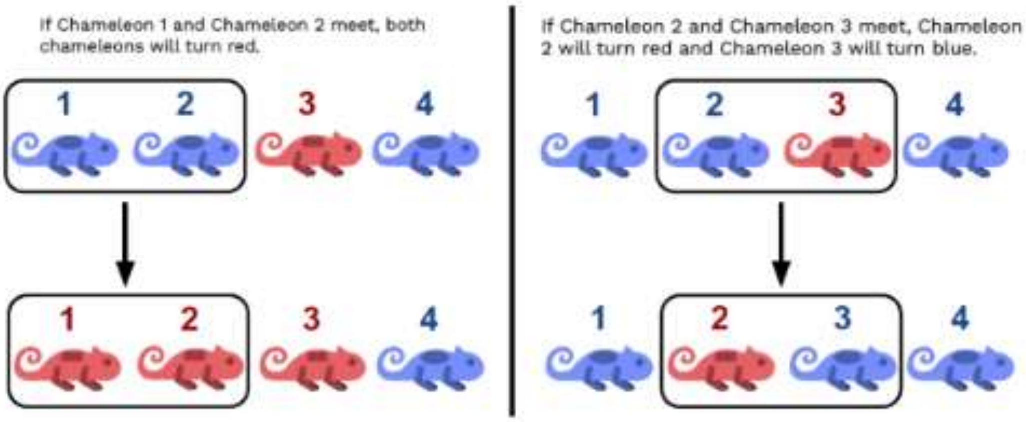
Examples of cooperative games are *Bandido*, *Max* (*the cat*) and *Hoot owl hoot* (type it into a search engine to find out about these games). In general, type in the search engine *cooperative games*, and you will meet a lot of such games. *The Human knot game* is also interesting. On link https://teambuilding.com/blog/human-knot you can find instructions as well as a link to the video.

Group games, where a team fights against a team, are also very good, because cooperation develops within the group. Football, basketball, ..., these are all games that develop a sense of space.

A4: Non-competitive games in which, according to some rules, some goal must be achieved. Here are some such games from the website https://jrmf.org/puzzle/ .

Chameleons: A number of blue and red chameleons are obtained and according to the rules below, all should be turned into red chameleons. Instead of using the application on the mentioned page, the game can be realized with red and blue chameleons cut out of paper.

When two chameleons meet, they both change colors, like this:



Spajanje posudica:

How to Play

Objective:

- End with all of the cups in a single stack.

Rules:

- Count the number of cups in a stack. That stack must jump that number of spaces. For example, 1 cup can only move 1 space; 2 cups have to move 2 spaces; 3 cups have to move 3 spaces...
- A cup or stack of cups cannot move into an empty space. They have to land on another cup or stack of cups.
- The game ends when no further moves are possible. If all of the cups are in a single stack, you win! If there are two or more stacks, try again!

A5: Frogs and Toads

How to Play

Objective:

- Can you help the frogs and toads get to the other side of the pond?

Rules:

There are seven rocks in a quarry with some frogs and toads on them As you can see below, there are three frogs on the left, three toads on the right, and an empty rock in the middle.

They would like to swap positions (toads on the left and frogs on the right).

To do so, the frogs and toads take turns moving one at a time in one of two different ways. Each turn, one frog or one toad can either:

Move one space forward onto an empty rock

OR

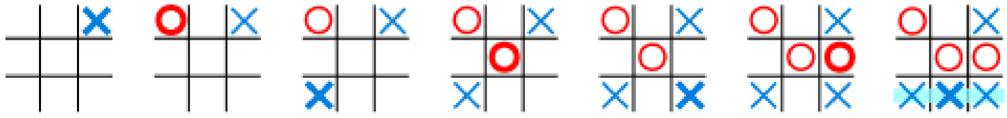
Jump over one other animal onto an empty rock.

A6: Closer – further game. An object is hidden, and the child is looking for it. If she goes closer to the object another child says *warm* or *closer*, and if she goes further from the object another child says *cold* or *further*. The game is good for acquiring the concept of distance.

Here are some classic competitive games.

A7: Hey, Don't Get Angry. It is a well-known game of chance (https://en.wikipedia.org/wiki/Mensch_%C3%A4rgere_Dich_nicht), where children learn to count and where they acquire the idea of chance, a very important mathematical idea. It is convenient to make these games in a large format, because then they are more impressive to children. Here's my attempt:

A8: Tic tac toe. A simple game to master the situation of making choices between multiple options with the goal of winning. The tic-tac-toe is played on an empty 3x3 square on paper. Player O places the circles and player X places the crosses. Starting with the player with the cross, players take turns selecting empty squares and drawing their mark inside. The player wins when they achieves 3 of his symbols in a row, column, main or side diagonal. If neither player succeeds, the game ends in a draw. Player X wins the following game:



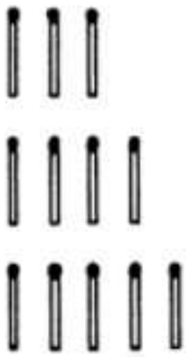
A9: Nim. It can be played with any number of rows and objects in a row.

Nim

A Game for Two

This is one of the oldest and most enjoyable of mathematical games for two. The word *nim* probably comes from the Shakespearean word meaning to steal. Possibly it was first played in China.

Nim is played with matches or coins. In the most popular version 12 matches are placed in three rows—3 matches, 4 matches, and 5 matches, as shown.



The rules are simple. The players take turns in removing one or more matches, but they must all come from the same row. The one who takes the last match wins. (You can also play the other way: The one to take the last match loses.)

Playing a few games will soon show you how you can always win: (*a*) Your move must leave two rows with more than 1 match in a row and the same number in each; (*b*) your move leaves 1 match in one row, 2 matches in the second row, and 3 in the third; or (*c*) if you play first, on your first move you take 2 matches from the top row and after that play according to the first two winning strategies just given.

You can play Nim with any number of matches or pennies in each row, and with any number of rows. As it happens, there is a way of working out how to take the right number of matches to get into a winning position. You simply use "computer counting," or binary. This method was first given in 1901. A description of it is given in the answers section.

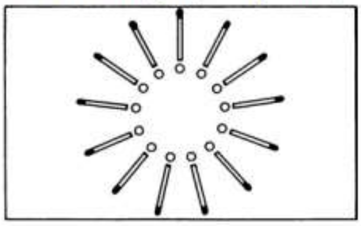
(the source of this text is unknown to me)

A10: Daisy.

Daisy
A Game for Two
The two players take turns to pluck from the daisy either one petal or two adjacent petals. The player taking the last petal is the winner. This is a game invented by the great puzzlist Sam Loyd.

Make a daisy with 13 petals out of matches, like this. On a postcard mark little circles where the petals (matches) grow from. You need to know whether you have left a space between petals or whether petals are next to each other. The second player can always win—if he knows how. See the answer section for this winning strategy.

Remember you cannot take two petals if there is a space between them. That's why we recommend marking the petals' positions.



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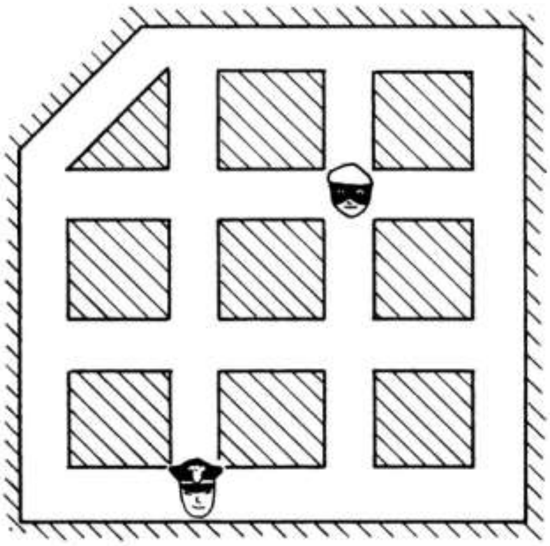
(the source of this text is unknown to me)

A11: The Cop and the Robber.

The Cop and the Robber
A Game for Two
Here is a single-board game for two. You can play on the city plan shown here or draw a larger version for yourself.



You need two coins, one for the cop, the other for the robber. Start with each coin on its picture. The rules are simple: The cop always moves first. After that, the players take turns to move. You move a coin one block only, left or right, up or down—that is, from one corner to the next. The aim is for the cop to catch the robber, which is done by the cop landing on the robber on his move. To make the game interesting, the cop must catch the robber in 20 moves, or he loses.

HINT: There *is* a way for the cop to nab the robber. The secret lies in the top left corner of the city plan.

(the source of this text is unknown to me)

A12: Chess https://en.wikipedia.org/wiki/Chess

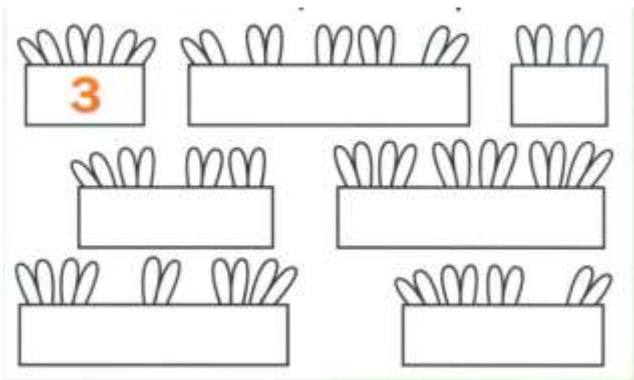
A13: Rubic's cube. https://en.wikipedia.org/wiki/Rubik%27s_Cube

Below you can find more games related to individual mathematical elements.

5 Numbers

Here, special care should be taken so that the child does not become too burdened with numbers, i.e. that activities with numbers are not too foreign to the child. You should choose activities for which the child is motivated, where the use of numbers is substantive and not formal, and which should not be too difficult for the child. More detailed instructions are given on the page 13.

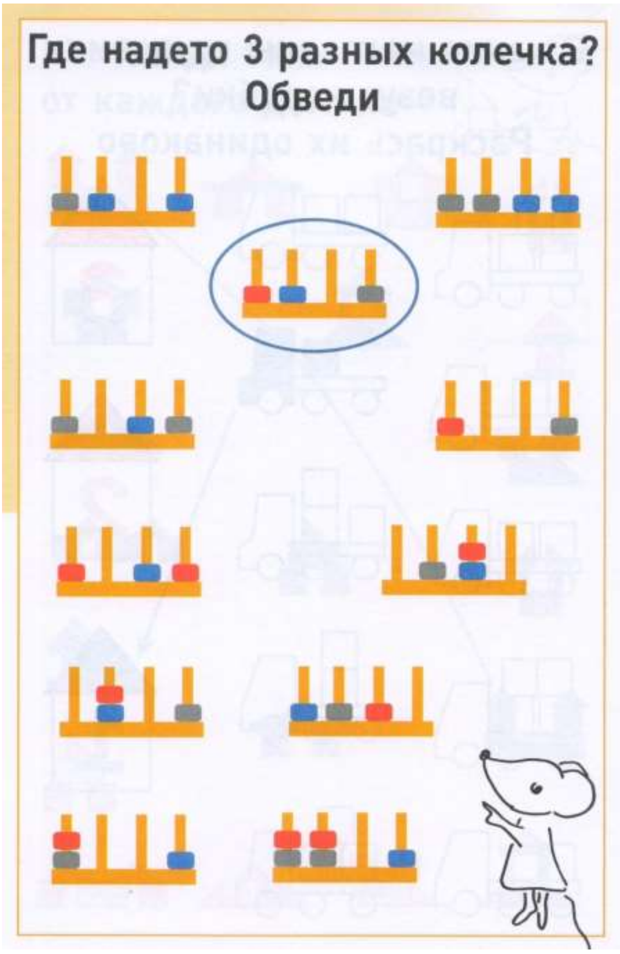
A1: Counting the members of a set. By counting, we measure how big a set is. Here are some interesting counts. How many rabbits are behind each fence?



(MouseMatics)

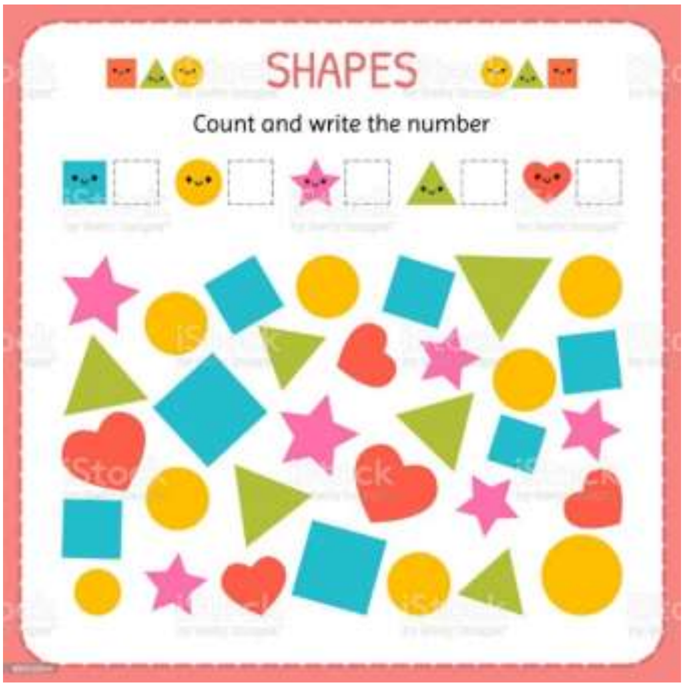
How many chickens are behind each fence?

(MouseMatics)



(MouseMatics)

A2: Comparison by counting. By counting, we compare which set has more items.



(taken from the internet, the site is unknown to me)

A3: Place in the sequence. Cross the third car and the fifth ball in the picture**:**

(Oštroumke, Element, Zagreb, 2021.)

A4: Counting songs. Like 10 little monkeys ili 5 little ducks: handy for counting down.

A5: Counting through stories. It is convenient to put a situation in the story where counting significantly affects the action. In Appendix 2 there is a sketch for one such story - a modification of the story *The Wolf and the Seven Little Goats*.

A6: Content-wise addition. Add two numbers using matches or some other available and easy-to-count objects. For example to determine how much 7+3 is, we take a pile of 7 matches, add 3 matches to the pile and count how many matches there are in total.

A7: Content-wise subtraction. Subtract two numbers using matches or some other available and easy-to-count objects. For example to determine how many are 7-3, we take a pile of 7 matches, take 3 matches from the pile and count how many matches are left.

A8: Formal addition and subtraction. This is a more advanced activity, so you need to carefully assess whether the child is interested in it. I will show my proposal on the example of 7+3 and 7-3. The child should first learn that in counting, moving to the next number is adding 1, and moving to the previous number is subtracting 1. Thus, moving from 7 to 8 is adding 1, and moving from 7 to 6 is subtracting 1. Adding 3 means adding 1 three times, and subtracting 3 means subtracting 1 three times. Thus, in order for a child to formally calculate 7+3, she should continue counting from 7: 8, 9, 10, 11, 12, ..., and know when to stop . She can do this by counting from 7 and 1 *at the same time*, so she says: 7 and 1 is 8, and 2 is 9 and 3 is 10. The same is true for subtraction: 7 less 1 is 6, less 2 is 5, less 3 is 4.

A9: Morr. *https://en.wikipedia.org/wiki/Morra_(game)*

Morra is an old game for two players. Players simultaneously show the outstretched fingers of one hand and try to guess the sum of the fingers shown, simultaneously shouting out a number ranging from 0 to 10. Although the game has elements of luck, the skill of the players comes to the fore.

A10: Cuisenaire rods. *https://en.wikipedia.org/wiki/Cuisenaire_rods/*. They are very convenient for comparing lengths, as well as for addition and subtraction. Teacher Simon Gregg blogs about lots of activities with Cuisenaire sticks: (https://followinglearning.blogspot.com/search?q=Cuisenaire)

A11: Konzervacija ukupnog broja predmeta. A certain number of items is displayed. One part is hidden. By counting the initial number of objects and the final number of objects, it is necessary to determine how many objects are hidden (subtraction)

A12: Noticing the difference by counting.

(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)

A13: Measurement by counting. Let's make a small worm out of paper. Crawling on the pencil, it counts how big the pencil is, crawling on the spoon, how big the spoon is, etc. On the *Internet Archive* website you can read and retell to children a wonderful story about a worm that measures - Lionna: *Inch by Inch*.

A14: Hunger measurer. The hunger measurer consists of, say, 10 biscuits lined up next to each other. A hungry child came and ate some biscuits. Using the final state of the hunger measurer, we can measure how many biscuits have been eaten (we add biscuits until the gladometer is full).

A15: Countdown to thunder. This is a good example of using counting. When we see lightning, we start counting at a normal pace, or practice a little by saying one number in one second. We count until we hear thunder. If we got to the number 6, it means that approximately 6 seconds have passed. Sound takes about 3 seconds to travel a kilometer. Thus, we can determine that the lightning was two kilometers away from us. By "counting" lightning, we can estimate whether a storm is approaching or receding. I taught my granddaughter Nina that if she counts less than 6 to the time of thunder (a distance of approximately 2 km), she must enter the house immediately.

A16: Counting forward and backward. Let's go along the road paved with slabs. If we don't step on the edge of the slab, we add 1, and if we do, we subtract 1. Naturally, a situation can arise here where we „fall“ below zero, so children can be motivated for negative numbers.

A17: Cooking. Preparing food according to a recipe is an ideal context for counting and measuring.

STEP 1

Put small piles of MKit Squares and MKit Pawns in front of your child. Have your child sort the squares and pawns by color.

STEP 2

Tell your child that that you are going to bake an MKit cake and that you need help. Tell your child to pretend that:

- is an egg
- is 1 cup of sugar
- is 1 cup of flour
- is 1 teaspoon of baking powder
- is a salt shaker

STEP 3

Get a bowl, a cup, or any other container that your child can put the "ingredients" in. Tell your child that the recipe calls for 2 eggs and ask your child to put the "egg squares" in a bowl. The recipe needs 1 cup of sugar, 2 cups of flour, a teaspoon of baking powder, and a shake of salt. Ask your child to put the ingredients in the container and stir them into "cake batter."

(the source is unknown to me)

A18: The idea of approximation. We try to guess „by eye“ how much of something there is.

A19: The idea of speed of change. We judge whether something is faster or slower than something else (e.g. cars on the road, or fast recitation of a text).

A20: The idea of dependency. At the most basic qualitative level, we can encourage the child to observe whether one quantity increases or decreases depending on the increase or decrease of another quantity. For example the pitch of the sound when we tap the glass with a spoon, depending on the amount of water in the glass.

A21: Multiplication and division. These operations should only be done with children if they naturally show interest, as happened to me with my granddaughter Nina, when she asked me to help her make 4 cardboard cars.

She asked me how many corks to bring. We found that we needed four times four corks, and she brought that many. I asked her how much four times four was, she counted the corks and said 16. Then she asked me how many toothpicks she needed. We found that we need half as many corks - let her take one toothpick for every two corks. When she collected the toothpicks, I asked her how much is half of 16. She counted the toothpicks and said 8. In an important context for a child, it naturally develops mathematics that appears in that context.

6 Geometry

You can briefly read about the importance of geometry in children's world on the page 15.

A1: Movement in space. The basic activity is moving and playing in a safe space and participating in spatial activities of adults.

(https://earlymaths.org/)

A2: Simulation of movement on drawings. The rocket, traveling from star to star, must reach our planet Earth:

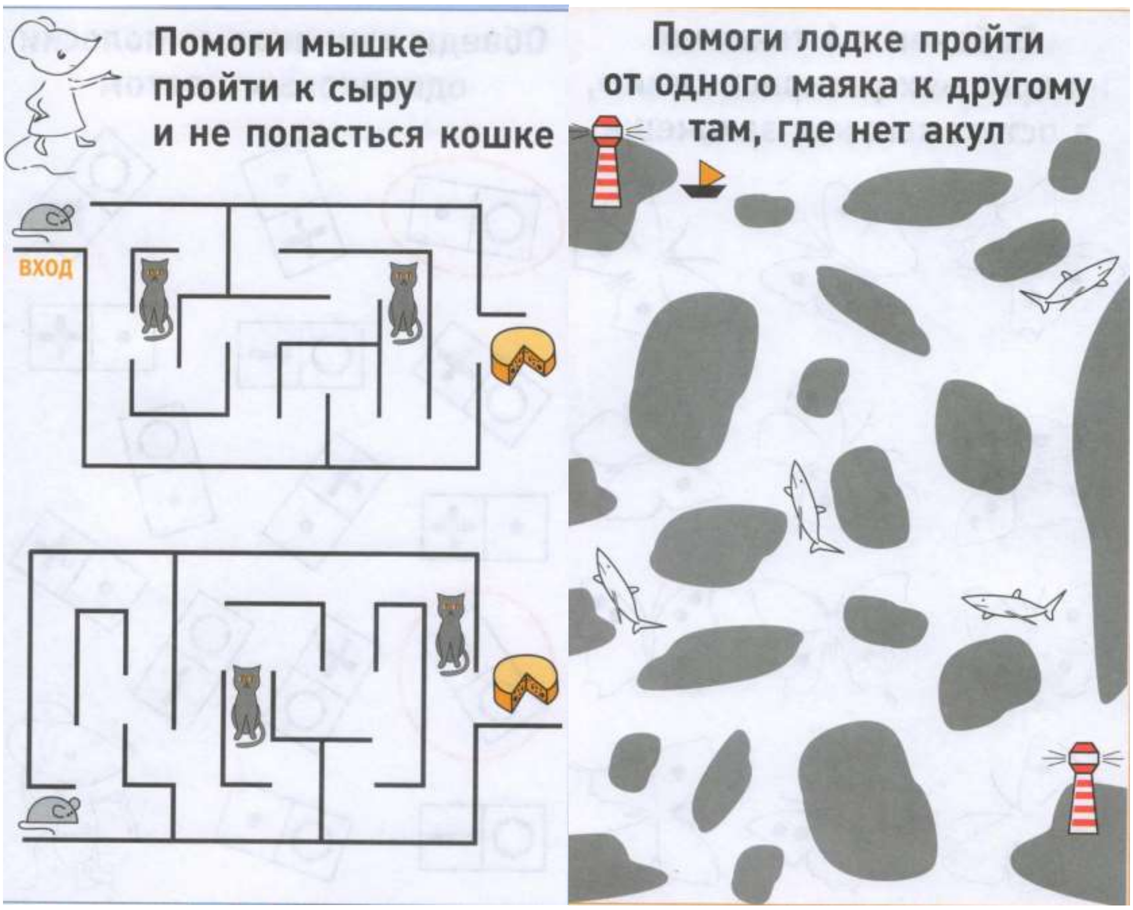
(taken from the internet, the site is unknown to me)

A3: Movement in an area with obstacles. Create a training ground or find a field in nature for such an activity.

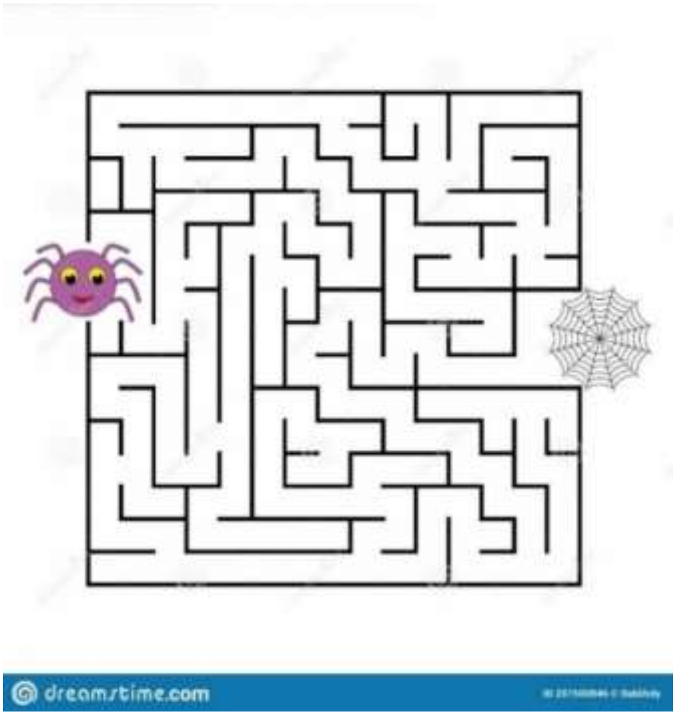
(https://earlymaths.org/)

A4: Hide-and-seek game.

A5: Labyrinths.



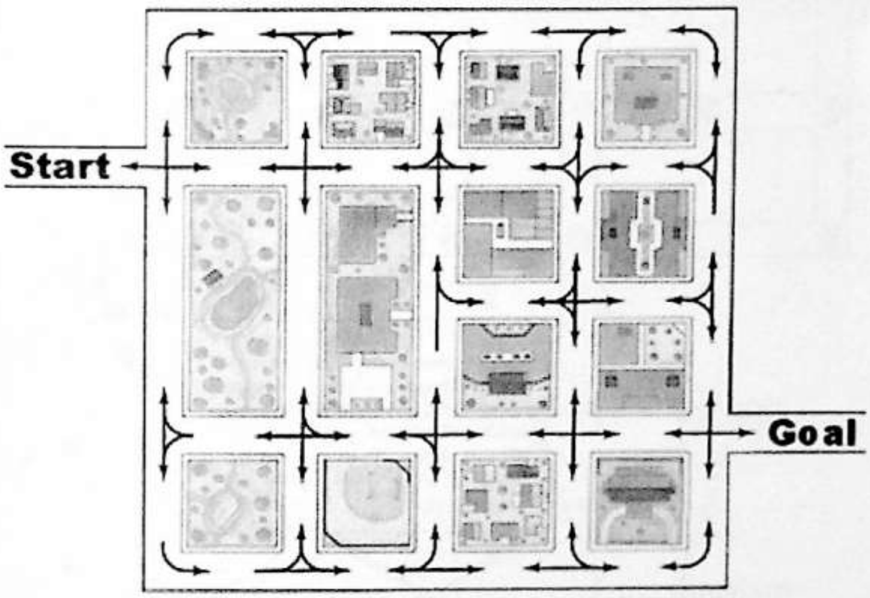
(MouseMatics)

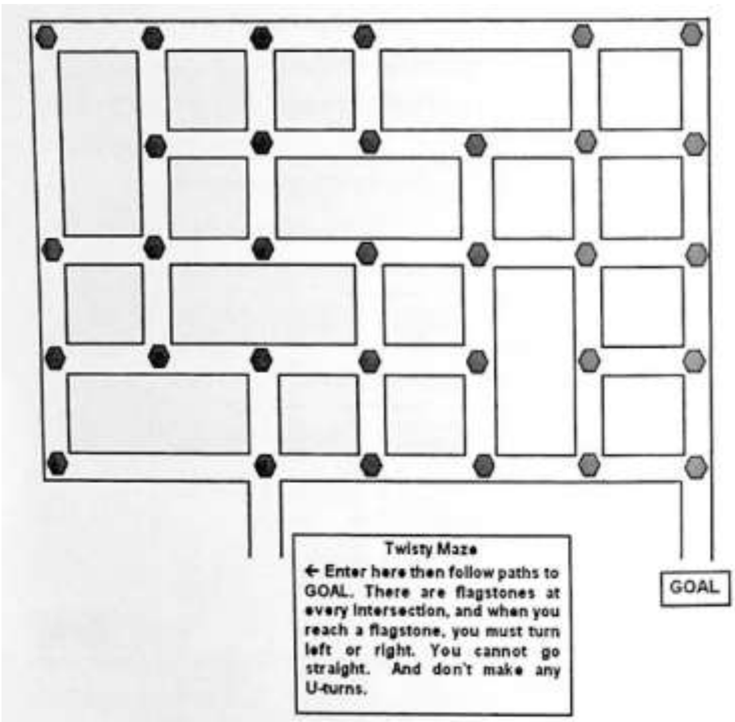


(taken from the internet, the site is unknown to me)

Here are some more complex examples:

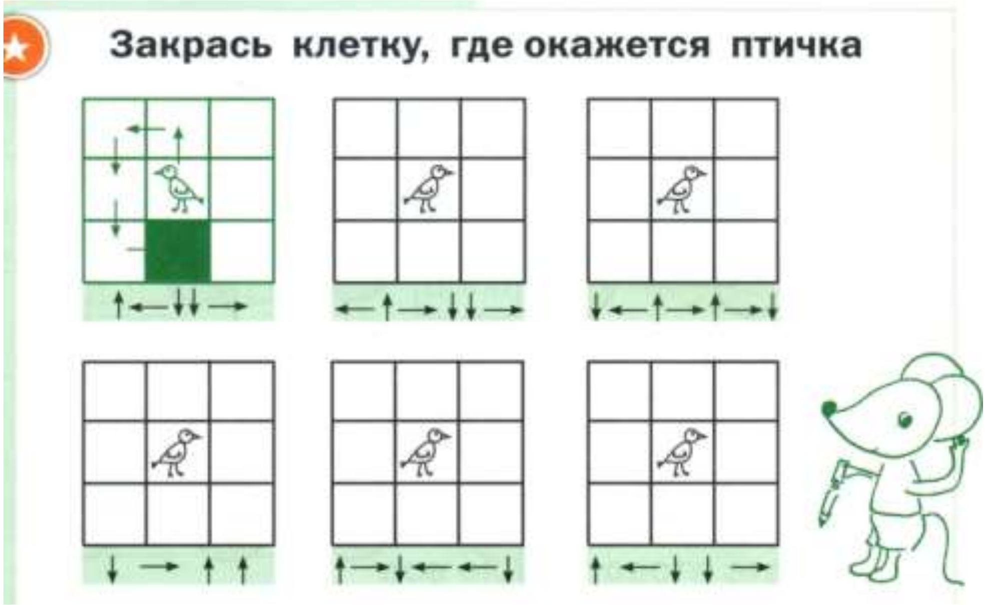
Travel along the roads from Start to Goal. At each intersection follow one of the arrows. That is, you can turn in a certain direction only when there is a curved line in that direction, and you can go straight only when there is a straight line. You can leave an intersection only at the head of an arrow. U-turns are not allowed.





You can find more such examples on the website http://www.logicmazes.com/

A6: Directions.



(MouseMatics)

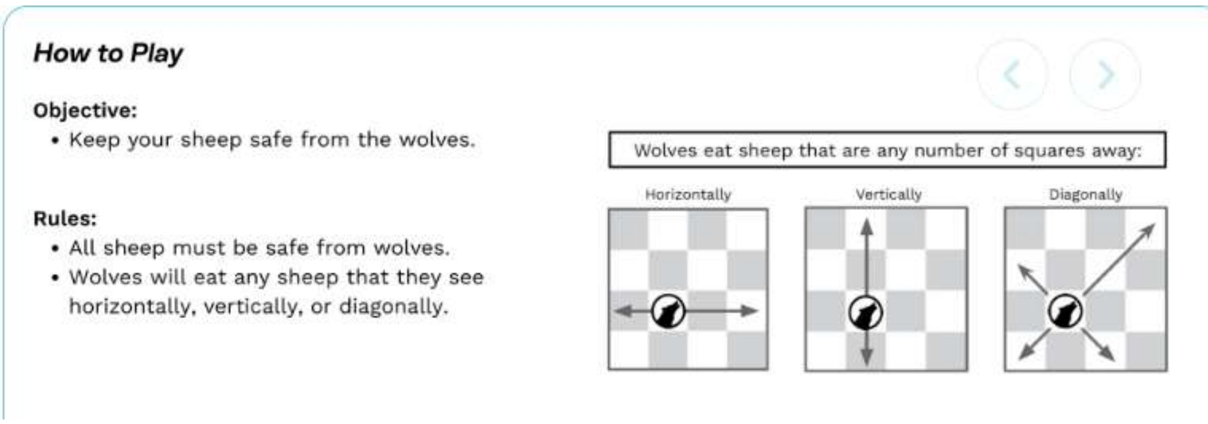
A7: Back scratching. Please scratch my back: up, down, left, right.

A8: Robot movement play. One child controls the robot (the other child) which executes the commands *go forward*, *go back, turn left, turn right, stop*. You can also enter how many steps go forward and backward. Using these commands, the robot should reach a certain place, avoiding obstacles.

A9: Distance estimation. Estimate which positions in space are further away, then measure in steps.

A10: Rotation Change directions in space and evaluate when we rotated more and when less and which way we rotated (children like to rotate).

A11: Wolfs and sheeps. A given number of sheep and wolves should be arranged on a given grid so that all sheep are safe.



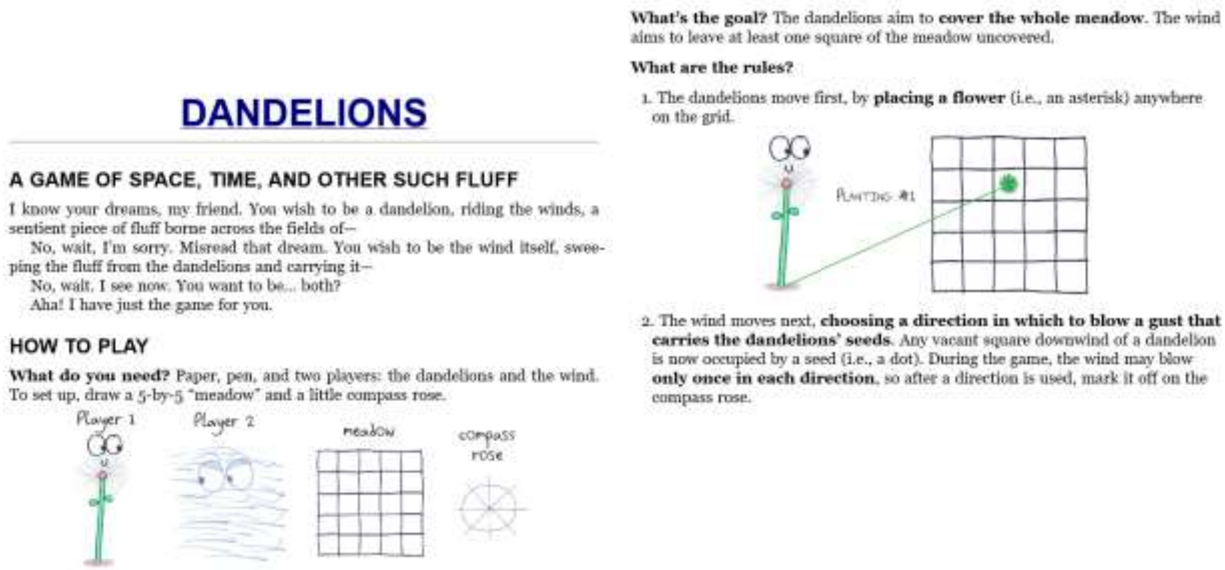
(the source is uknown to me)

Nina's solution:

A12 Spatial relations.

(Oštroumke, Element, Zagreb, 2021.)

A13: Game with directions - Dandelions. (from the book Ben Orlin: *Math Games with Bad Drawings*)



DANDELIONS

A GAME OF SPACE, TIME, AND OTHER SUCH FLUFF

I know your dreams, my friend. You wish to be a dandelion, riding the winds, a sentient piece of fluff borne across the fields of—

No, wait, I'm sorry. Misread that dream. You wish to be the wind itself, sweeping the fluff from the dandelions and carrying it—

No, wait. I see now. You want to be... both?

Aha! I have just the game for you.

HOW TO PLAY

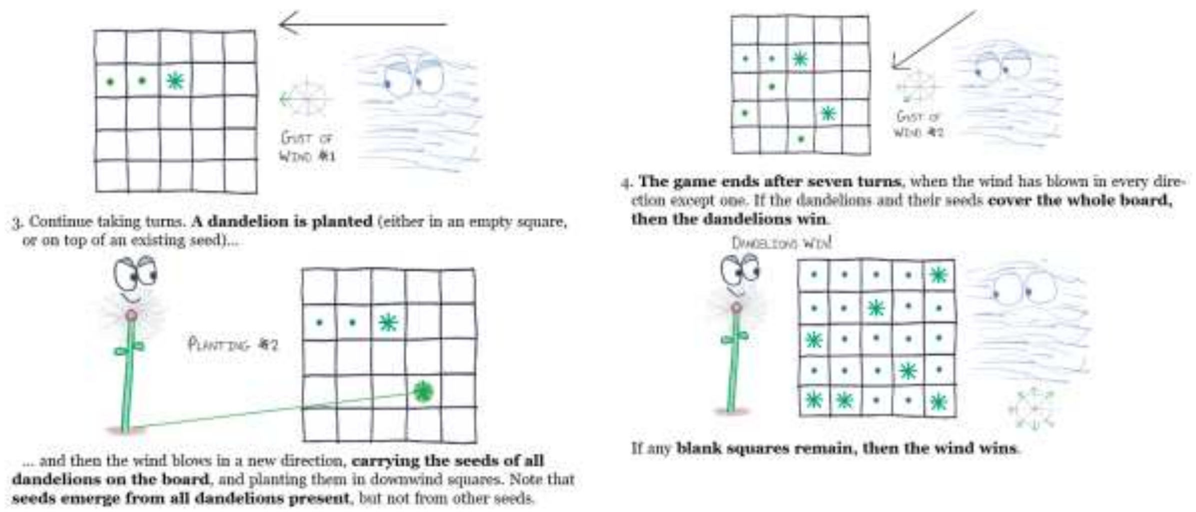
What do you need? Paper, pen, and two players: the dandelions and the wind. To set up, draw a 5-by-5 "meadow" and a little compass rose.

What's the goal? The dandelions aim to **cover the whole meadow**. The wind aims to leave at least one square of the meadow uncovered.

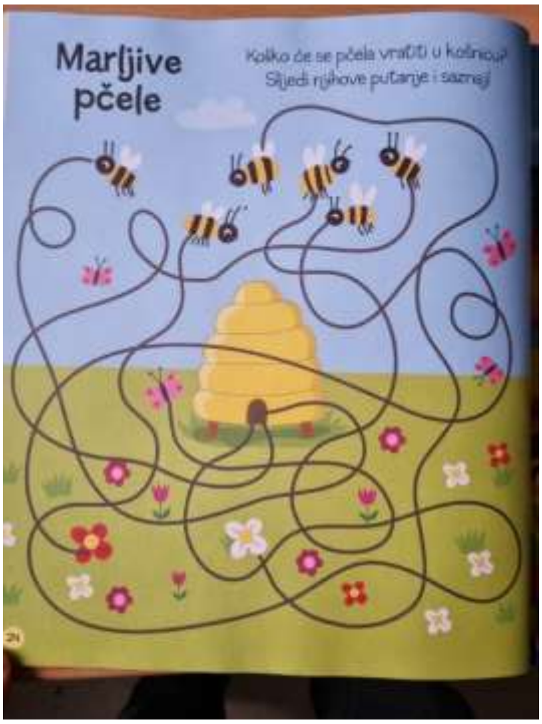
What are the rules?

1. The dandelions move first, by **placing a flower** (i.e., an asterisk) anywhere on the grid.

2. The wind moves next, **choosing a direction in which to blow a gust that carries the dandelions' seeds**. Any vacant square downwind of a dandelion is now occupied by a seed (i.e., a dot). During the game, the wind may blow **only once in each direction**, so after a direction is used, mark it off on the compass rose.



A14: Line tracking.

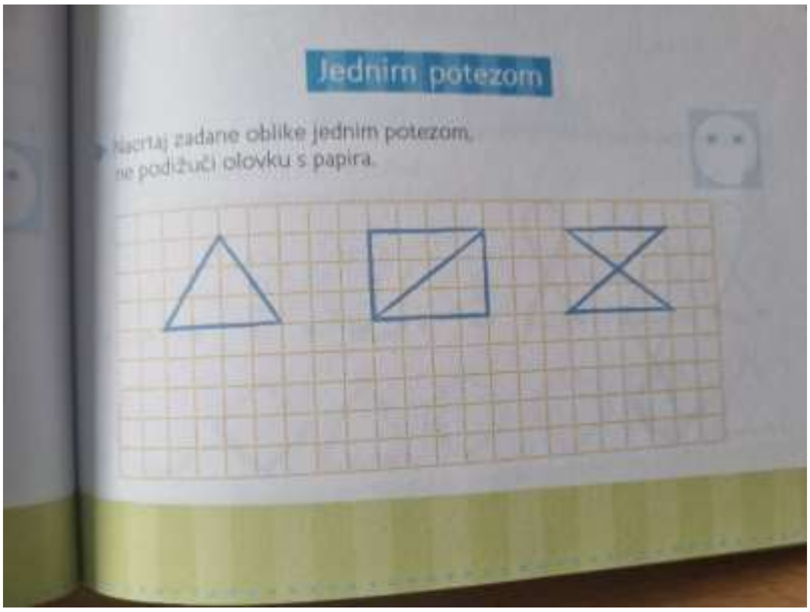


(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)

A15: Drawing directions, lines and arrows. Drawing on paper, drawing in the sand, etc.

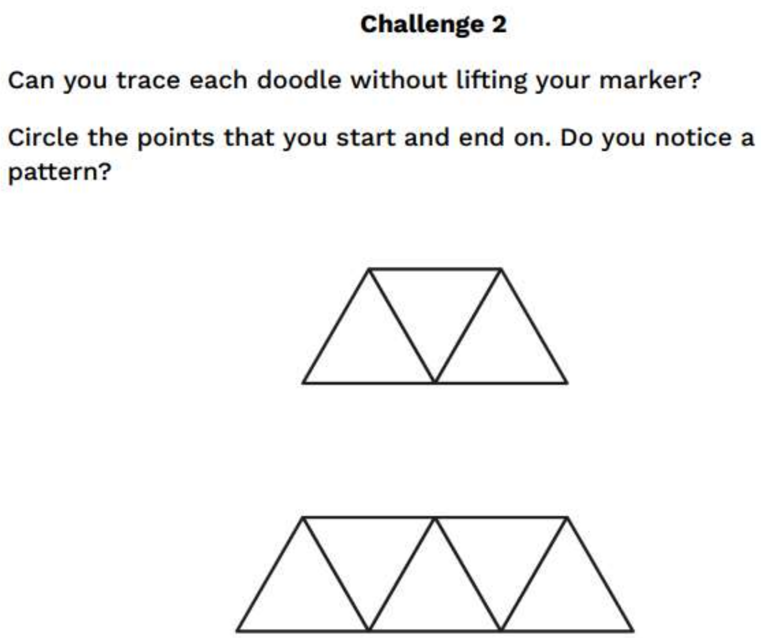
A16 Connecting two points. Between two points, draw a straight line, a broken line, a curved line. Which is the shortest? Is there a longest line? Notice that straight lines are the shortest connectors.

A17: Euler walks. We start from one place and, without lifting the pencil, we go through each line of the character exactly once and return to the beginning (this is not always possible - it is possible just when an even number of paths meet at each „crossroads“).



(Dejić: Matematika kao igra)[25]

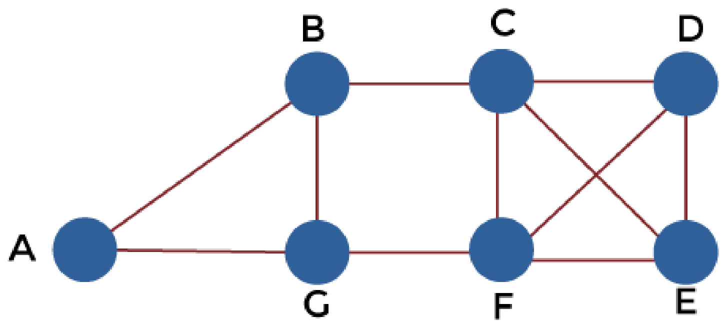
A slightly different task – we don't have to return to the same point.



(the source is uknown to me)

[25] Branka Dejić, Mirko Dejić: Matematika kao igra 1 i 2: Element, 2015

A18: Paths in a graph. It is convenient to draw graphs on the floor instead of on paper.



(taken from the internet, the site is unknown to me)

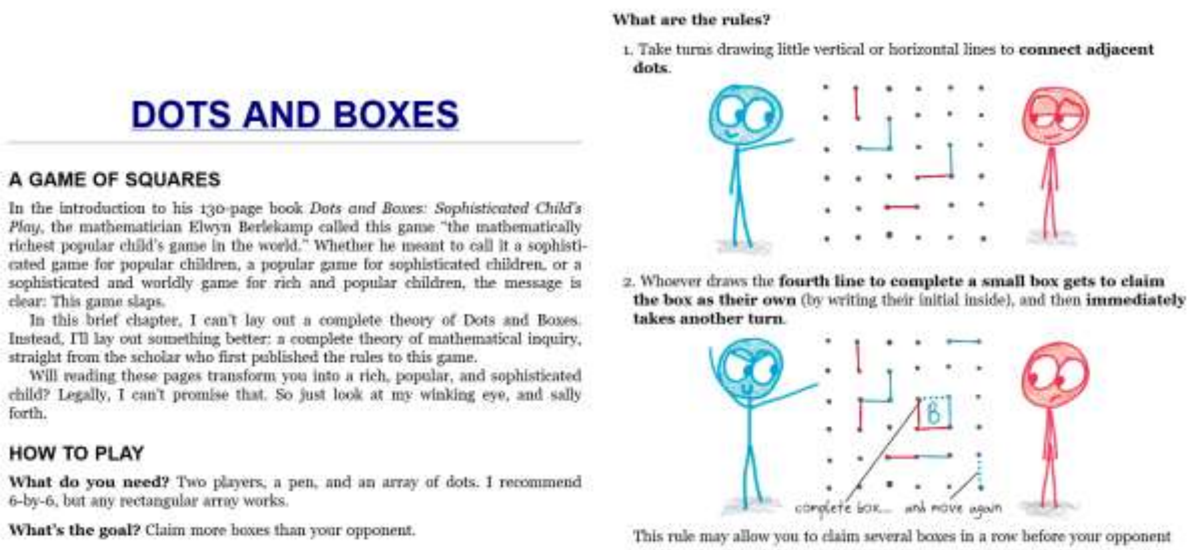
Lines can also have arrows on them. The lines can be given a length, but then those would be tasks for schoolchildren. A path consists of lines and we will consider it to be as long as the lines it contains. Lines are also called *edges* of the graph, and meeting points of the edges are *nodes* of the graph.

The shortest path problem: Take various paths in the graph from point B to point E. Which path is the shortest?

The bread delivery man's problem: Start from place A (your bakery), visit all nodes (shops that sell your bread) and return to place A. Find the shortest such path.

The problem of a street cleaner: Start from place A (your street maintenance company), clear all the streets (lines) and return to place A. Find the shortest such path.

A19: Game with lines (iz knjige Ben Orlin: *Math Games with Bad Drawings*)



DOTS AND BOXES

A GAME OF SQUARES

In the introduction to his 130-page book *Dots and Boxes: Sophisticated Child's Play*, the mathematician Elwyn Berlekamp called this game "the mathematically richest popular child's game in the world." Whether he meant to call it a sophisticated game for popular children, a popular game for sophisticated children, or a sophisticated and worldly game for rich and popular children, the message is clear: This game slaps.

In this brief chapter, I can't lay out a complete theory of Dots and Boxes. Instead, I'll lay out something better: a complete theory of mathematical inquiry, straight from the scholar who first published the rules to this game.

Will reading these pages transform you into a rich, popular, and sophisticated child? Legally, I can't promise that. So just look at my winking eye, and sally forth.

HOW TO PLAY

What do you need? Two players, a pen, and an array of dots. I recommend 6-by-6, but any rectangular array works.

What's the goal? Claim more boxes than your opponent.

What are the rules?

1. Take turns drawing little vertical or horizontal lines to **connect adjacent dots**.

2. Whoever draws the **fourth line to complete a small box gets to claim the box as their own** (by writing their initial inside), and then **immediately takes another turn**.

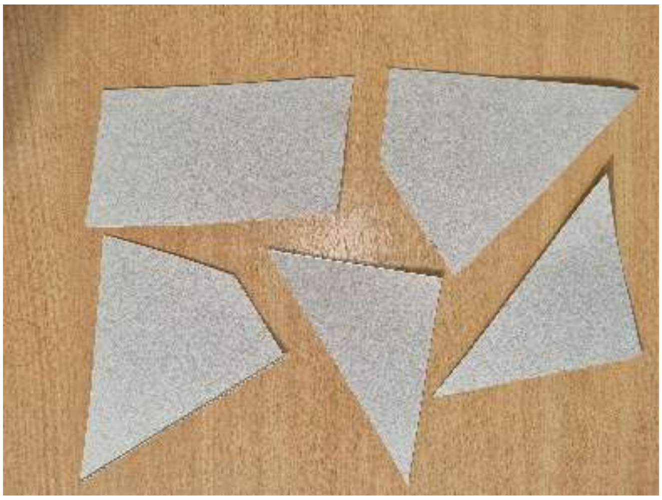
This rule may allow you to claim several boxes in a row before your opponent

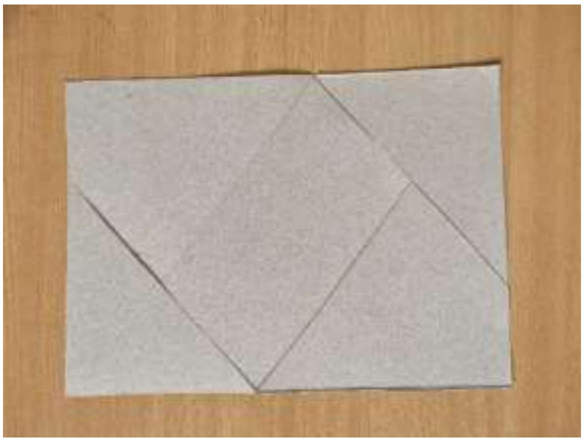
A19: Constructions in space. Various construction materials: wooden blocks, Lego, magnets, etc.

A20: Shaping. In clay, dough, sand, etc.

A21: Construction in plane. Using various 2d construction elements

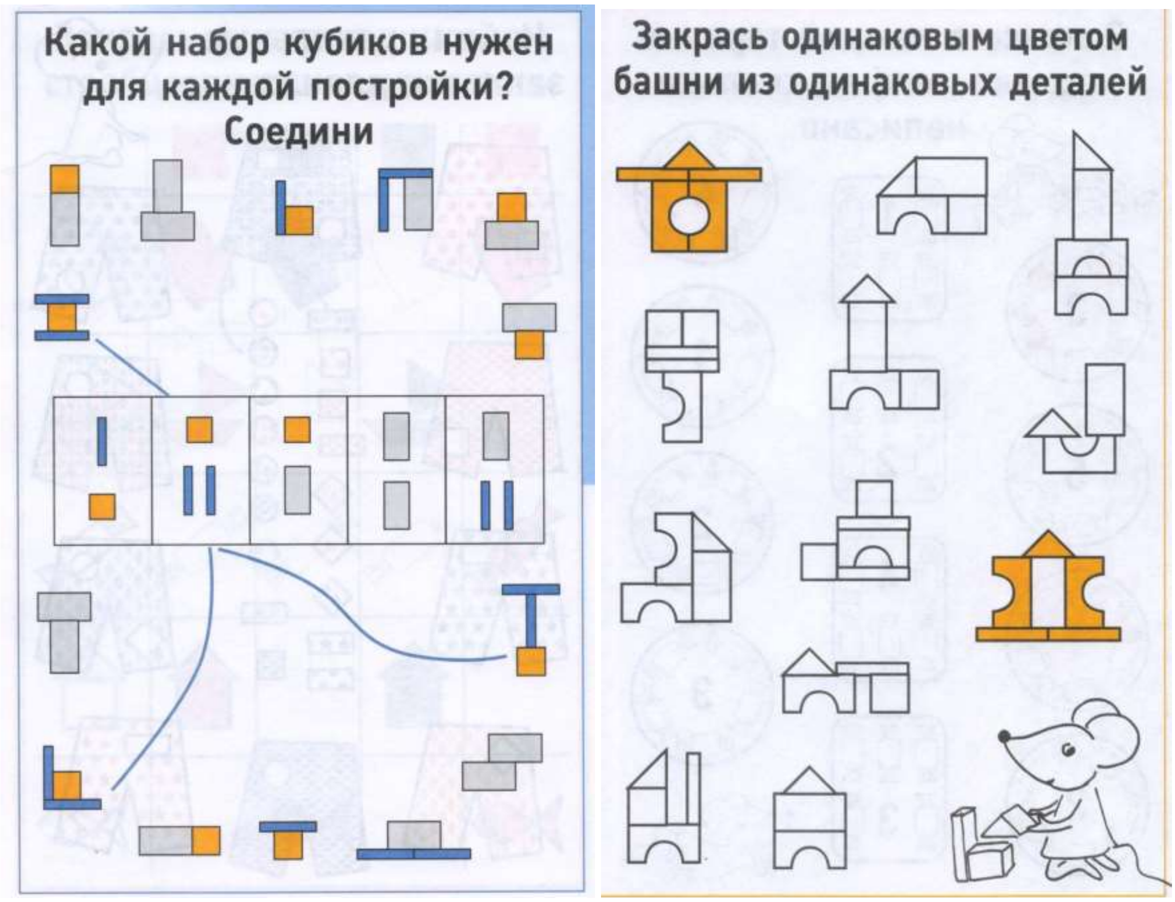
A22: Connecting parts. Purchased puzzles as well as do-it-yourself puzzles can be used for this. For example, cardboard is cut into pieces and joined together, first with pictures (easier) and then without pictures (harder):





A23 Disassembly into parts.

(MouseMatics)

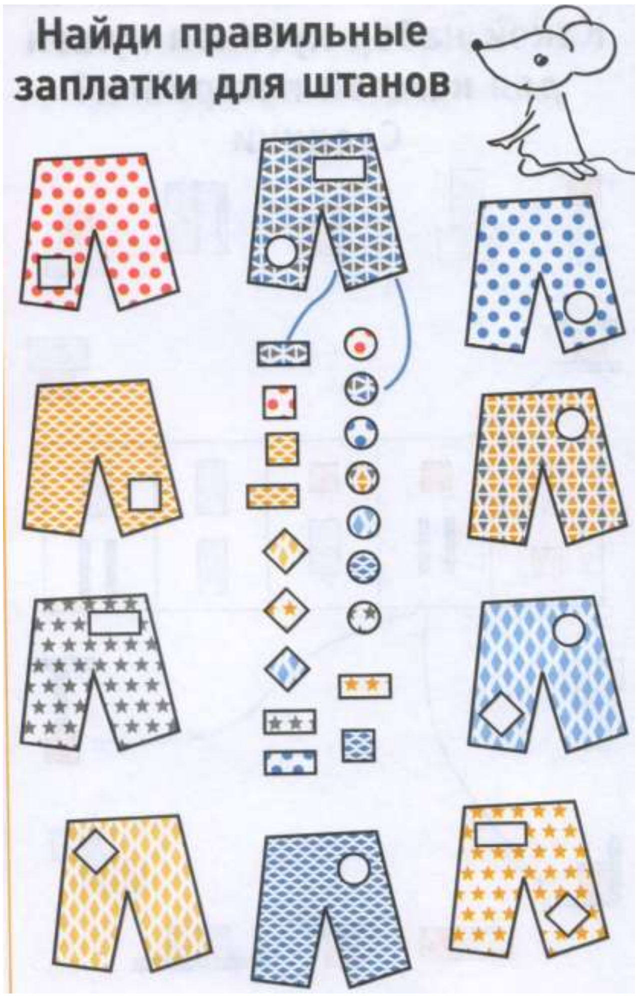


(MouseMatics)

(MouseMatics)

A24: Fill it out.

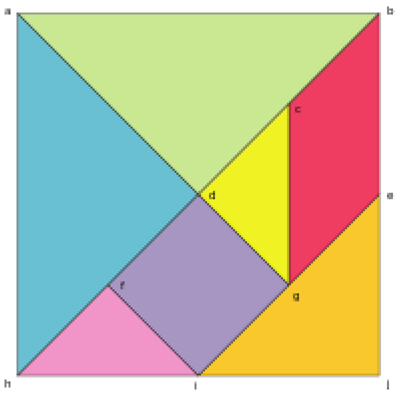
(MouseMatics)

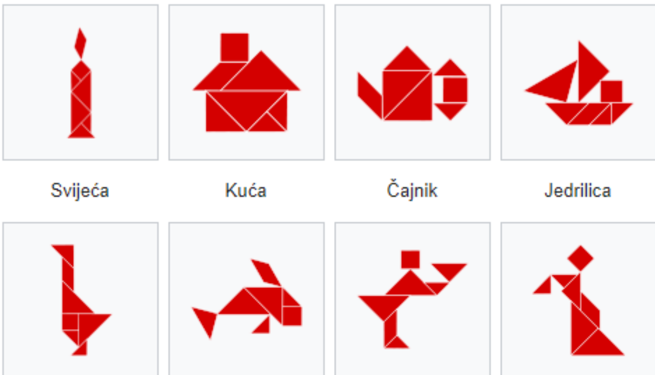


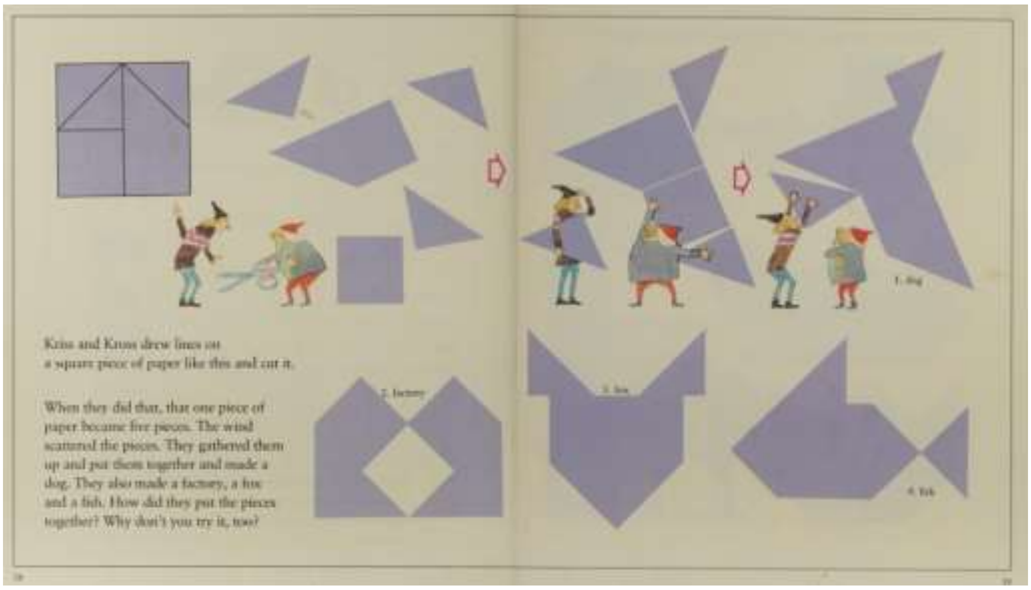
(MouseMatics)

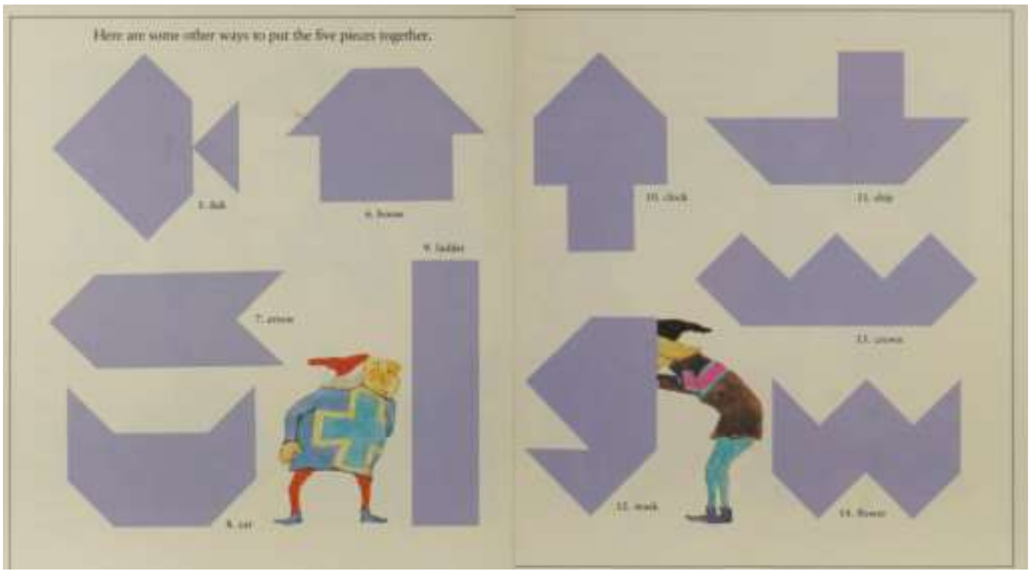
A25: Assembling figures.

A26: Making figures from parts. Various puzzle sets can be found on the market. Tangram is an old and extremely good puzzle. It consists of 7 parts from which all kinds of figures can be made (https://hr.wikipedia.org/wiki/Tangram)

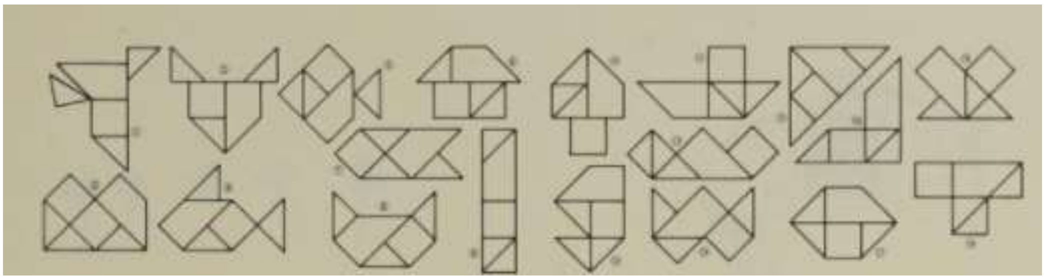






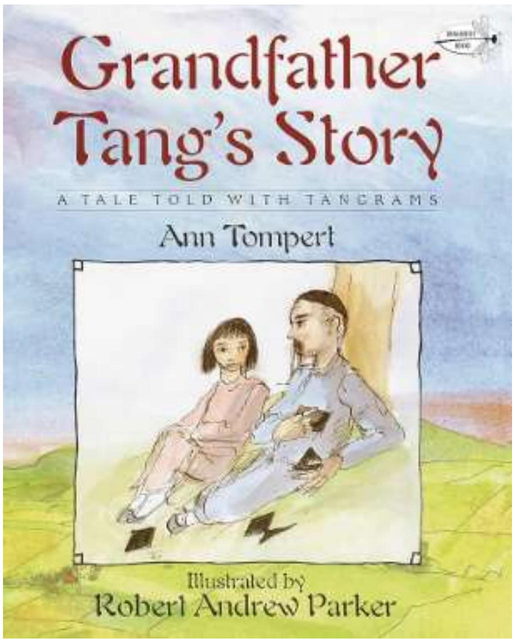


Here are the solutions for the figures above:



(Anno: Mathe Games I)

A27: Grandfather Tang's Story. You can find the story on the *Internet Archive* website (type the title into the search field). Make two tangrams and try to reproduce the story yourself. It would be ideal for you to assemble one tangram and the child to assemble the other.



A28 Moving parts.

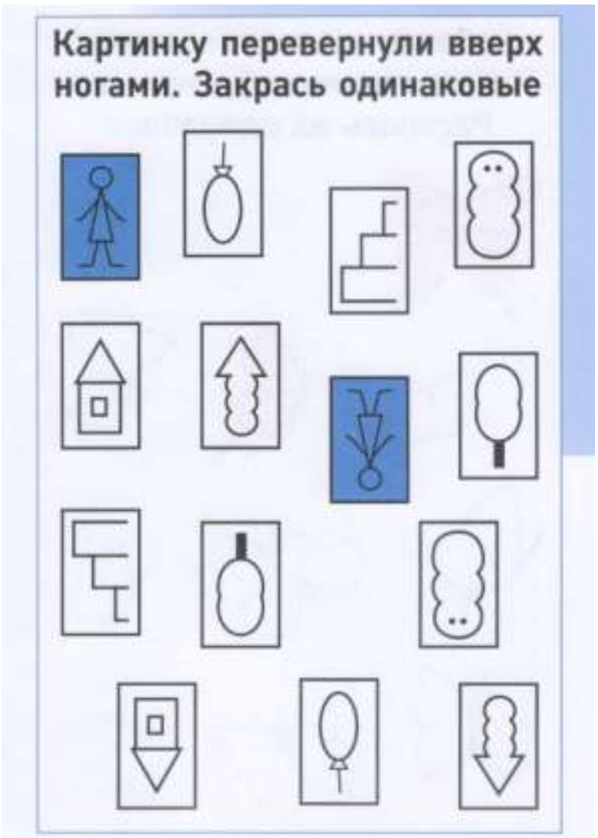
(MouseMatics)

(MouseMatics)

A29 Rotation.

(MouseMatics)

A30 Turnover.

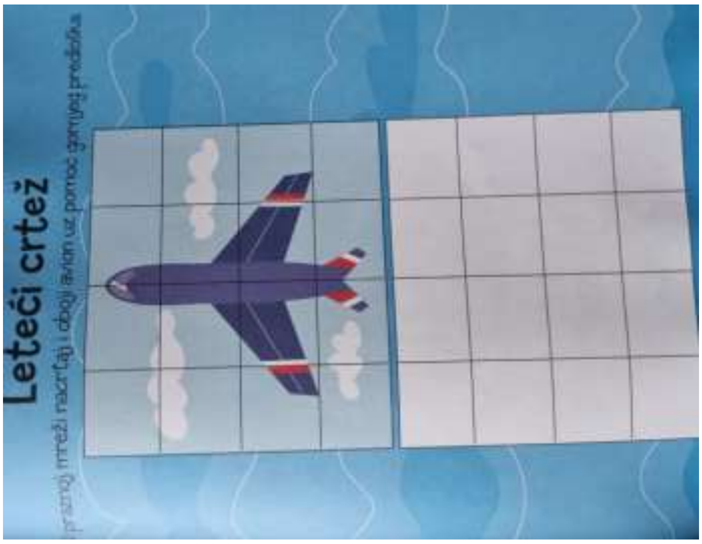


(MouseMatics)

A31: Drawing. Any drawing is welcome. This develops motor skills, a sense of shapes, the skill of representing a situation, whether spatial or emotional. Each drawing is a small mathematical model of something. Here is Nina's Neanderthals model (after visiting Krapina Neanderthal Museum)

A32: Copying figures

(MouseMatics)

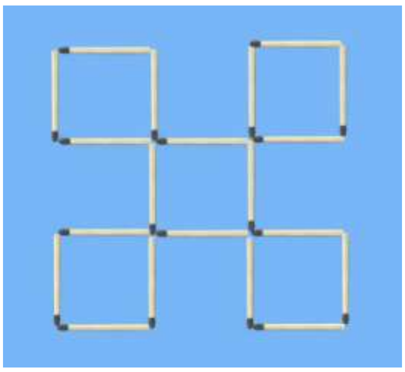


(*Fora zbirka Super zadataka, Naša Djeca, Zagreb*)

A33: Drawing a rotated figure.

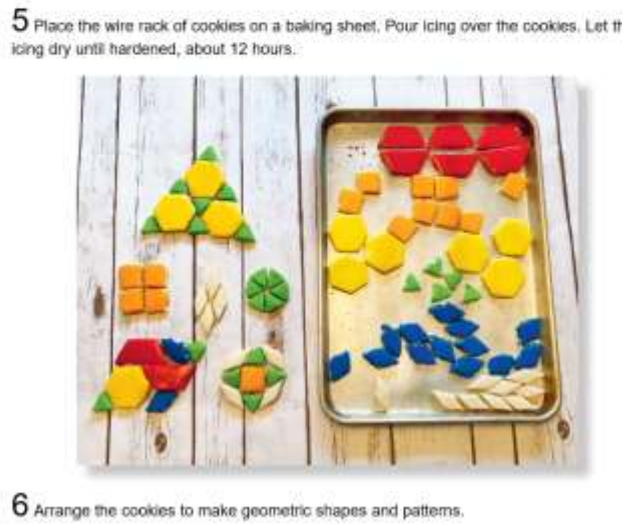
(MouseMatics)

A34: Making figure. It is interesting for children to make various figures with their own fingers. Figures can also be made from matches:



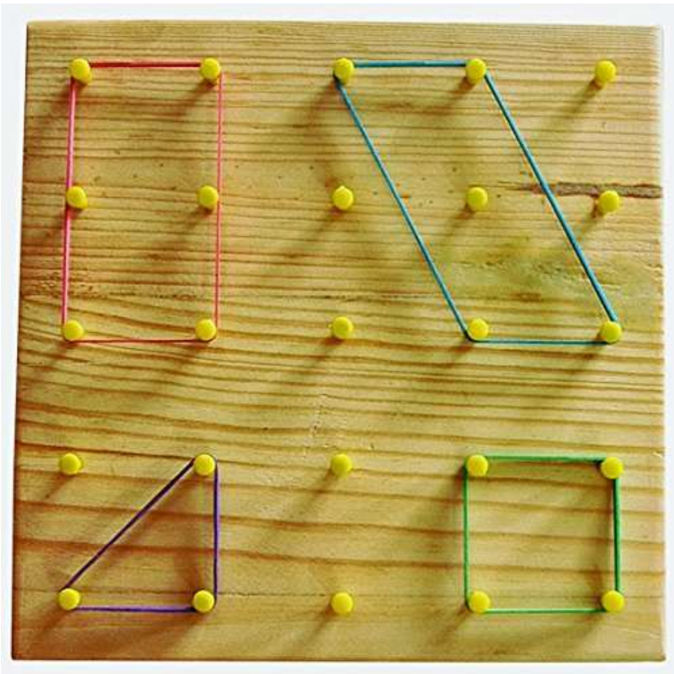
(https://suresolv.com/brain-teaser/)

They can also be baked in the oven!



(from the book Karyn Tripp: *Math Art + Drawing Games for Kids*)

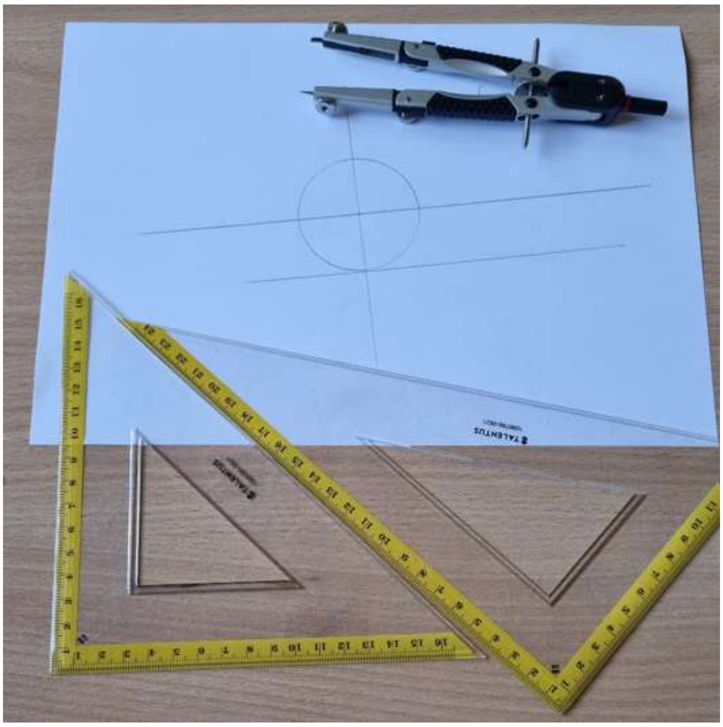
Geoboard can also be used, purchased or you can make it yourself on a board by driving nails at regular intervals. Figures are made by stretching rubber bands.



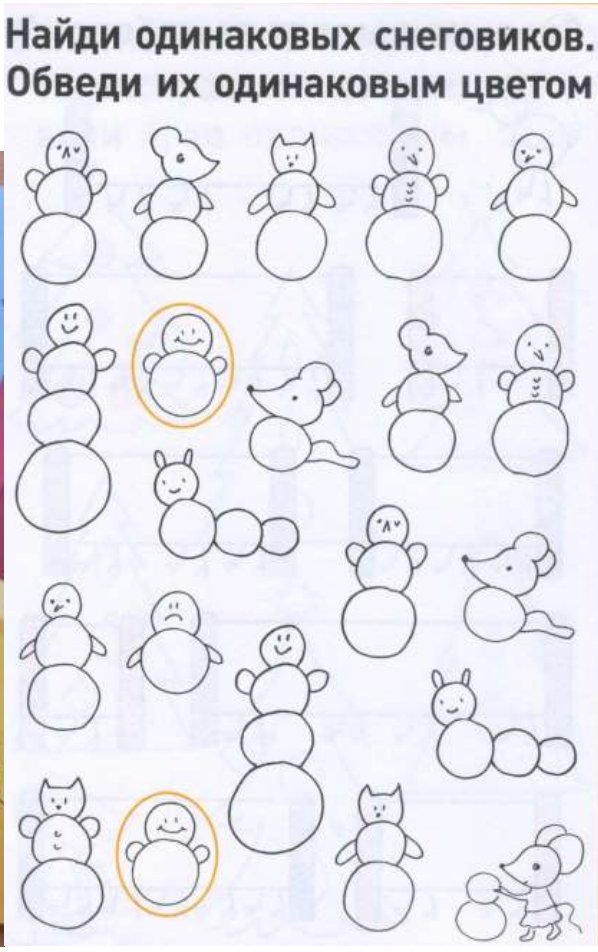
(https://www.amazon.in/IDEK-Math-Wood-Geoboard/dp/B00LCH0WWQ)

You can find various ideas for making figure on the geoboard on the website https://classplayground.com/geoboard/

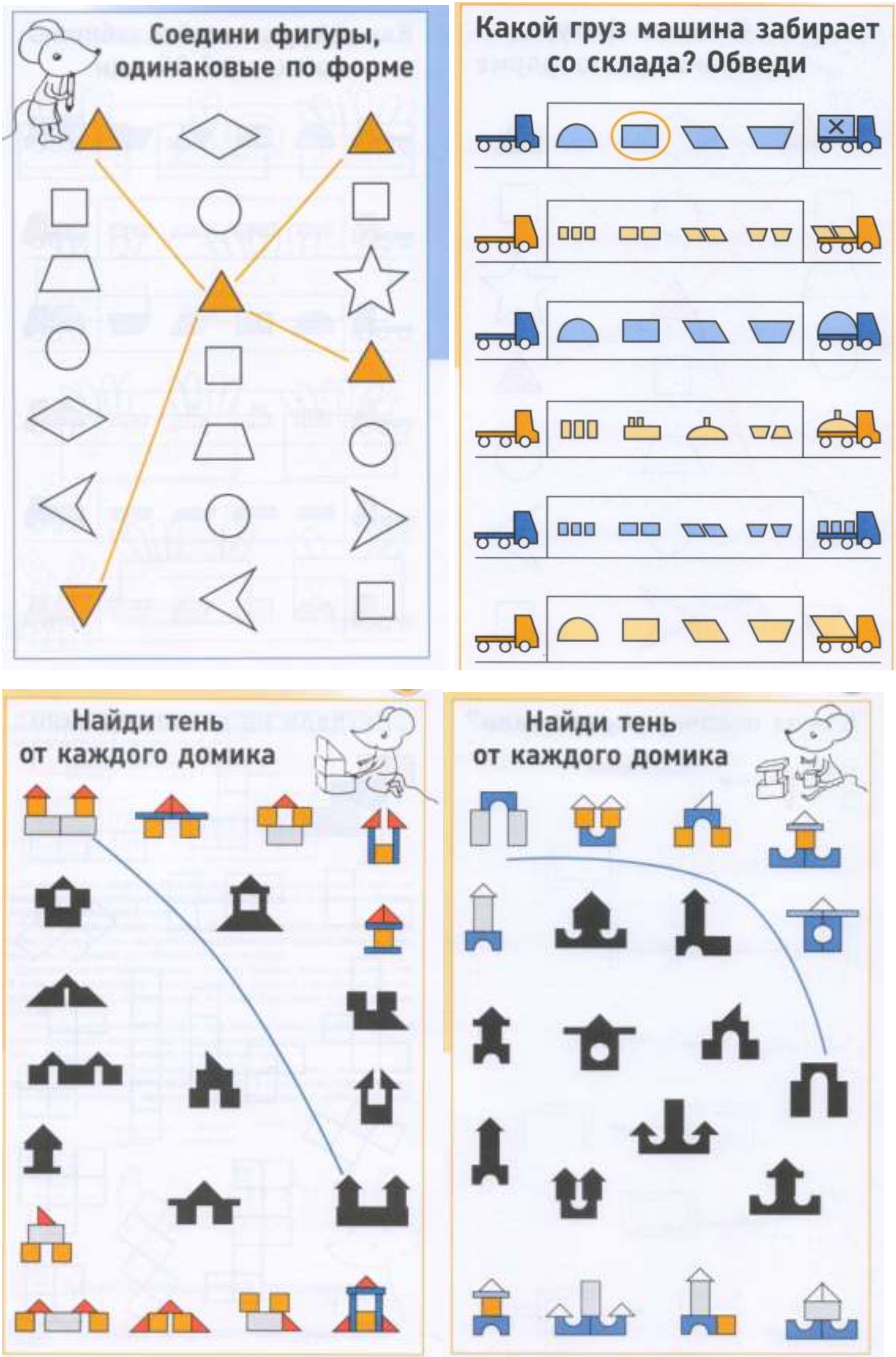
Kids can even practice making stright line and circles using rulers, triangles and compasses:



A35: Shape recognition.

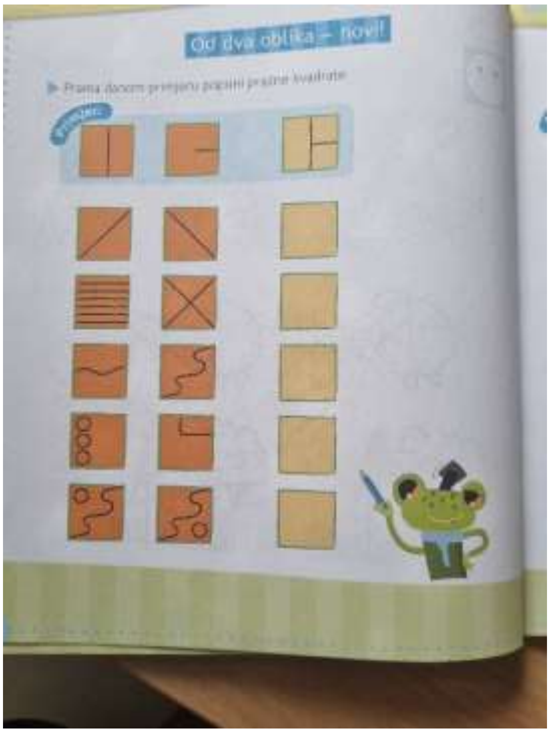


(Fora zbirka zadataka) (MouseMatics)



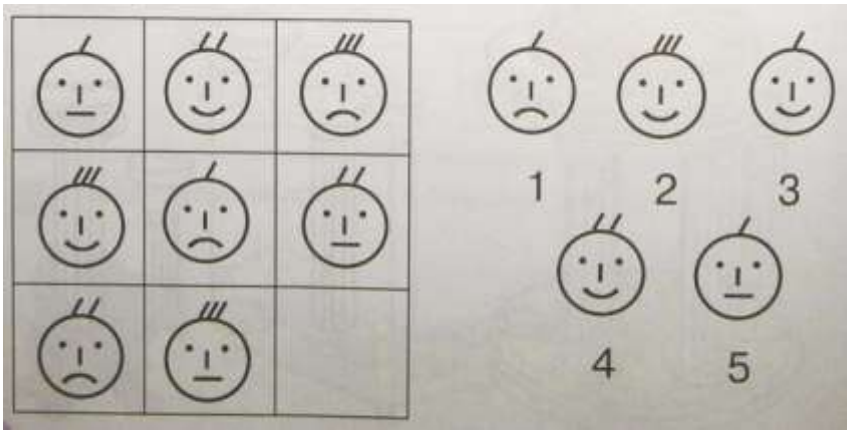
(MouseMatics)

(MouseMatics)

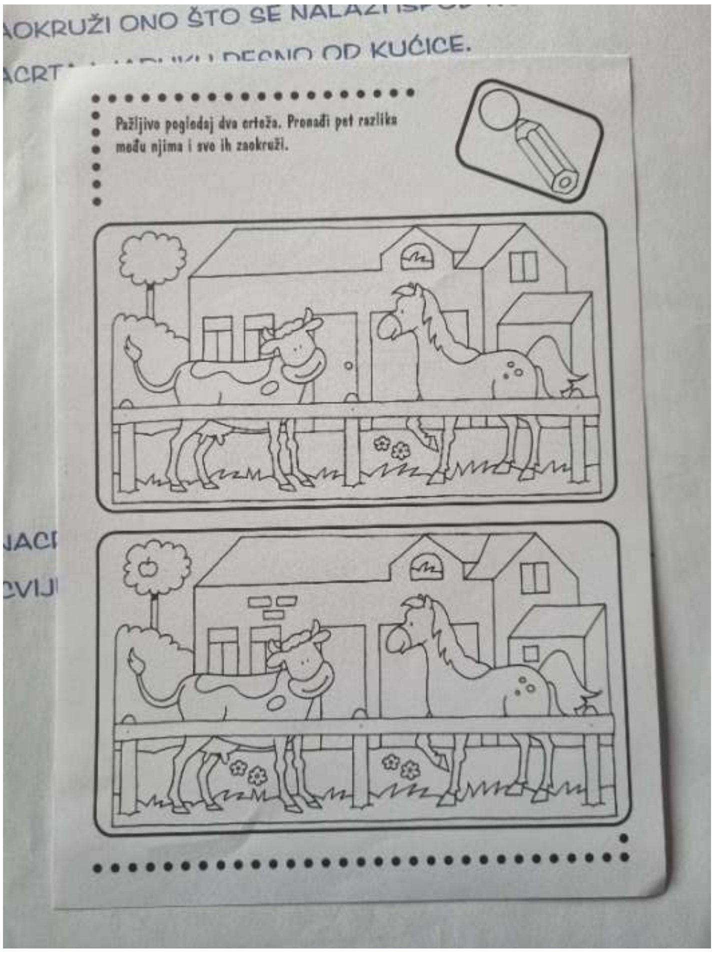


(Dejić: Matematika kao igra)

A36: Spotting differences:



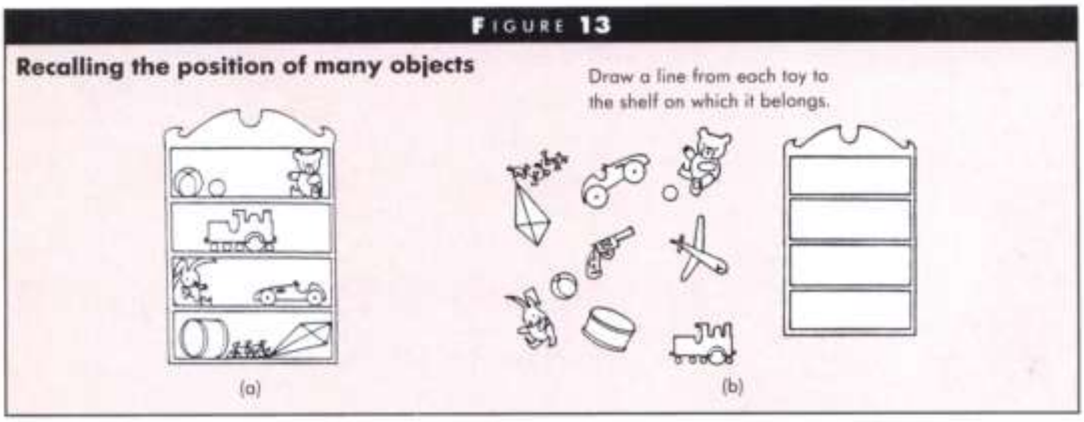
(the source is unknown to me)



(the source is unknown to me)

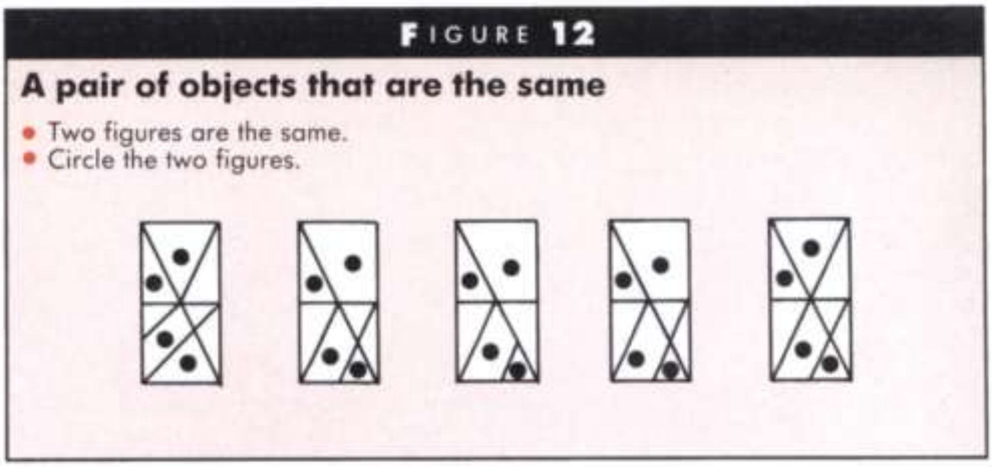
A37: Spatial arrangement of things. Space planning, room arrangement, wardrobe arrangement, arrangement of shelves with toys.

A38: Describing a spatial situation. Game: You look at some space, and you have to guess where what is by memory.

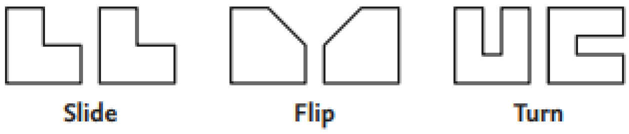


(the source is unknown to me)

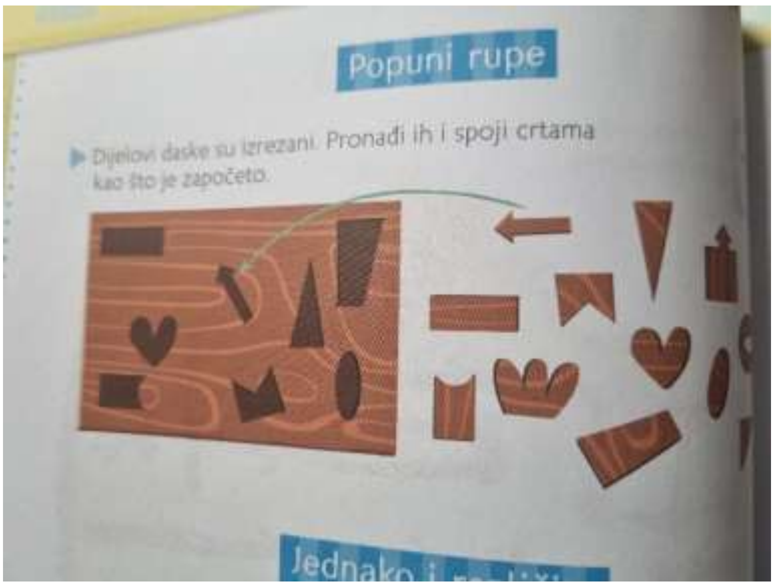
A39: Congruence and similarity. Spotting congruent figures - figures that can be moved and overlapped (have the same shape and dimensions). Spotting similar figures - figures that have the same shape (by enlarging and moving one figure can be superimposed on another).



(nepoznat mi je izvor)



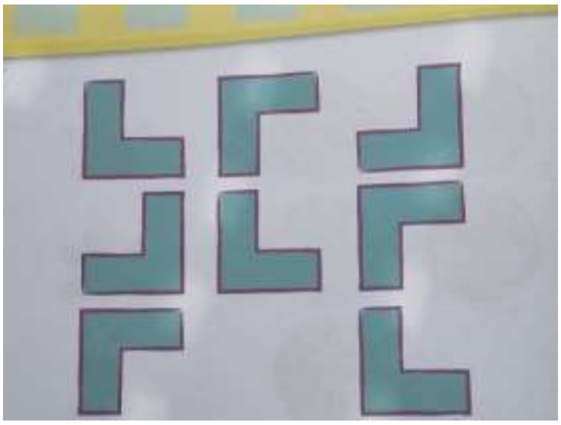
(the source is unknown to me)



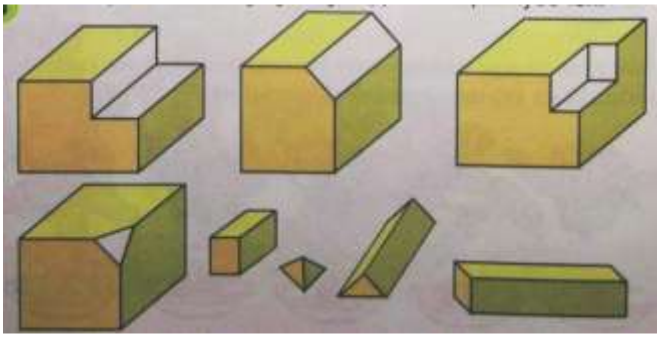
(Dejić: Matematika kao igra)

(Dejić: Matematika kao igra)

Fill in the missing figure:



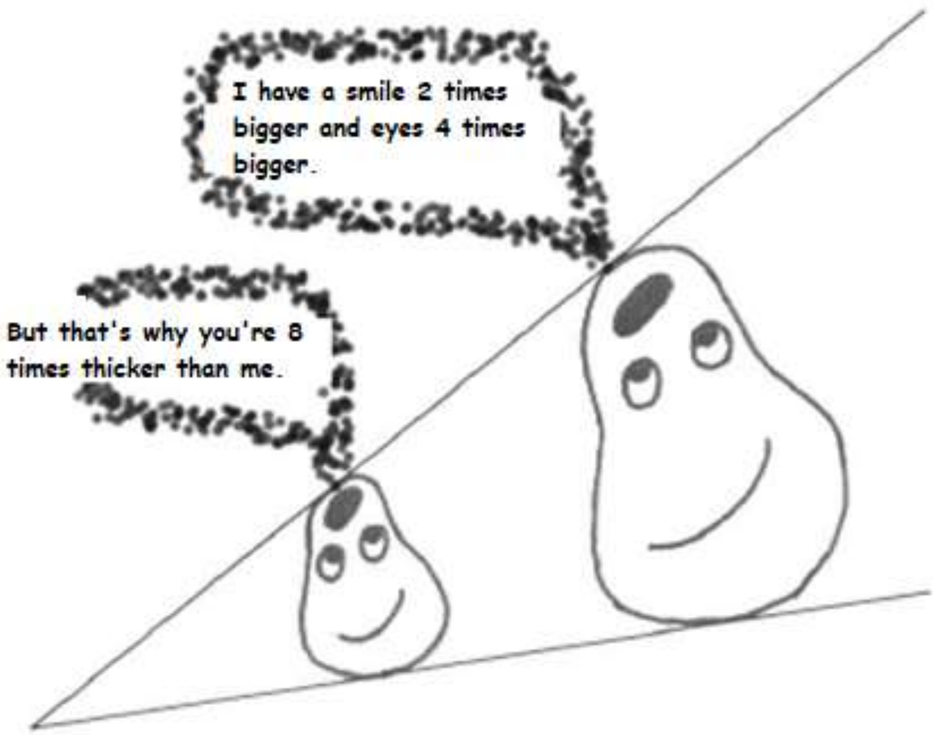
(Dejić: Matematika kao igra)



(Mousematics)

(*Fora zbirka Super zadataka*)

A40: Scaling. Drawing a reduced or enlarged figure.



A41: Symmetries. Behind our sense of symmetry lies a precise mathematical concept. The symmetry of a figure is the transformation of the plane (or space) that transforms the figure into itself. The following pictures show the different types of symmetries beautifully. Unfortunately, I lost the information from which book I copied these pictures.

MIRROR, MIRROR...

Look at your reflection in a mirror, it is exactly the same as you, except it is back to front. When a shape is reflected it might look very different or exactly the same, it depends on where you put the mirror. If it looks the same, the mirror is on a **line of symmetry**.

HALVE IT!
You might be able to draw a face, but can you draw just half?

1 Draw a straight line down the center of a piece of paper. This is your line of symmetry where you will place a mirror after you have finished drawing.

2 Draw half the face on just one side of the line.

3 It might help to fold the paper in half along the line you have drawn so you don't accidentally draw on the wrong side.

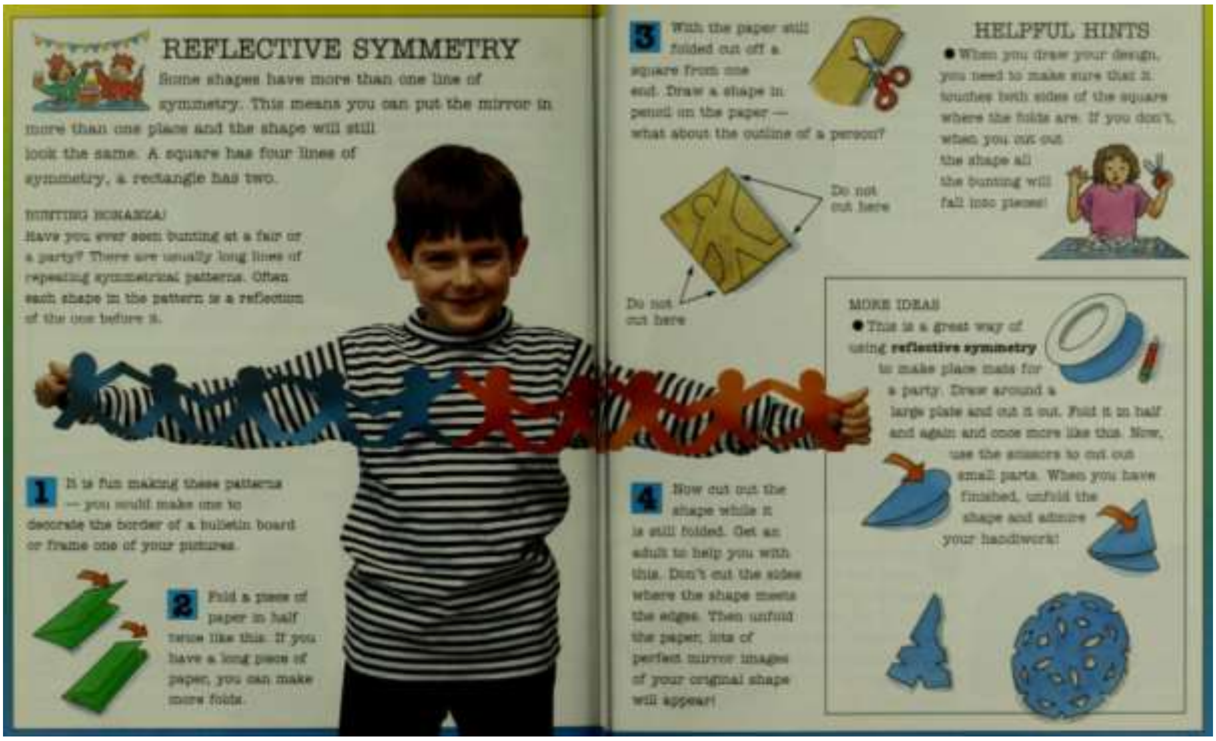
4 When you have finished your picture, put a mirror along the line of symmetry and look into it. You will see a face with two sides that are exactly the same, making a whole symmetrical face.

HELPFUL HINTS

- Before you start to draw the face, look at your own face in a mirror. Hold a large book up across half your face like this. How many eyes can you see? Ears? How much of your mouth can you see? What about your nose?

MORE IDEAS

- A famous Italian inventor and artist named Leonardo da Vinci used to keep a diary. He wanted to keep his thoughts secret so he wrote it in mirror writing, back to front! It was hard to read.
- Can you write your name in mirror writing? Try it and check by holding it up to a mirror.



REFLECTIVE SYMMETRY

Some shapes have more than one line of symmetry. This means you can put the mirror in more than one place and the shape will still look the same. A square has four lines of symmetry, a rectangle has two.

BUNTING BONANZA!
Have you ever seen bunting at a fair or a party? There are usually long lines of repeating symmetrical patterns. Often each shape in the pattern is a reflection of the one before it.

1 It is fun making these patterns — you could make one to decorate the border of a bulletin board or frame one of your pictures.

2 Fold a piece of paper in half twice like this. If you have a long piece of paper, you can make more folds.

3 With the paper still folded cut off a square from one end. Draw a shape in pencil on the paper — what about the outline of a person?

Do not cut here

Do not cut here

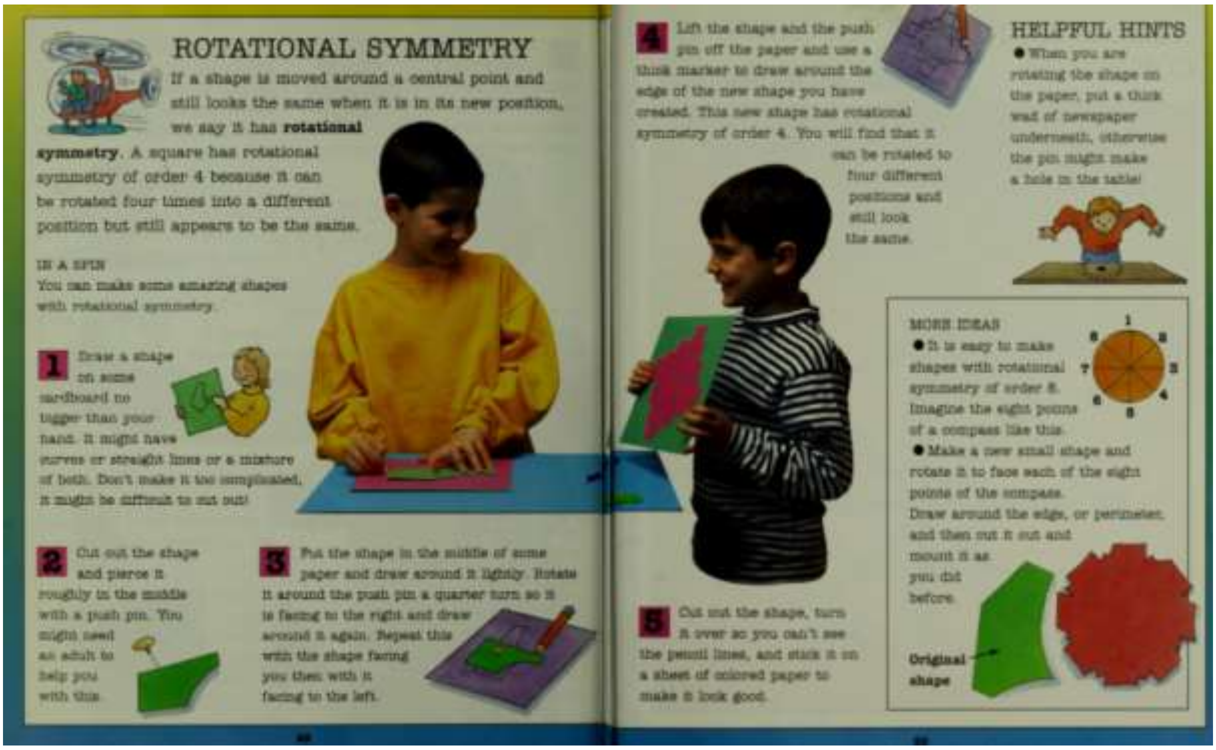
4 Now cut out the shape while it is still folded. Get an adult to help you with this. Don't cut the sides where the shape meets the edges. Then unfold the paper; lots of perfect mirror images of your original shape will appear!

HELPFUL HINTS

- When you draw your design, you need to make sure that it touches both sides of the square where the folds are. If you don't, when you cut out the shape all the bunting will fall into pieces!

MORE IDEAS

- This is a great way of using **reflective symmetry** to make place mats for a party. Draw around a large plate and cut it out. Fold it in half and again and once more like this. Now, use the scissors to cut out small parts. When you have finished, unfold the shape and admire your handiwork!



ROTATIONAL SYMMETRY

If a shape is moved around a central point and still looks the same when it is in its new position, we say it has **rotational symmetry**. A square has rotational symmetry of order 4 because it can be rotated four times into a different position but still appears to be the same.

IN A SPIN

You can make some amazing shapes with rotational symmetry.

1 Draw a shape on some cardboard no bigger than your hand. It might have curves or straight lines or a mixture of both. Don't make it too complicated, it might be difficult to cut out!

2 Cut out the shape and pierce it roughly in the middle with a push pin. You might need an adult to help you with this.

3 Put the shape in the middle of some paper and draw around it lightly. Rotate it around the push pin a quarter turn so it is facing to the right and draw around it again. Repeat this with the shape facing you then with it facing to the left.

4 Lift the shape and the push pin off the paper and use a thick marker to draw around the edge of the new shape you have created. This new shape has rotational symmetry of order 4. You will find that it can be rotated to four different positions and still look the same.

5 Cut out the shape, turn it over so you can't see the pencil lines, and stick it on a sheet of colored paper to make it look good.

HELPFUL HINTS

- When you are rotating the shape on the paper, put a thick wad of newspaper underneath, otherwise the pin might make a hole in the table!

MORE IDEAS

- It is easy to make shapes with rotational symmetry of order 8. Imagine the eight points of a compass like this.
- Make a new small shape and rotate it to face each of the eight points of the compass.

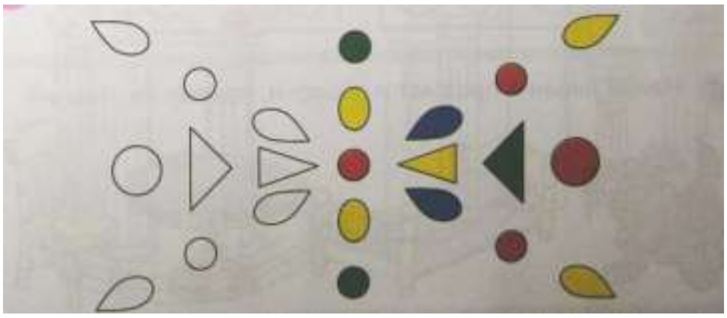
Draw around the edge, or perimeter, and then cut it out and mount it as you did before.

Original shape

A42: Symmetries by drawing.

(Dejić: Matematika kao igra)

(Fora zbirka zadataka)



(taken from the internet, the site is unknown to me)

Trace the shape. Turn it a small amount and trace it again. Continue turning and tracing the way around until you reach the beginning.

(Tripp: *Math Art and Drawing Games for Kids*)

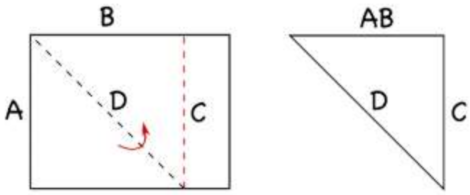
A43: Symmetries with paper and scissors.

Very interesting symmetrical figures can be obtained by folding and cutting paper. There are tons of videos and websites on the internet that demonstrate how to do it. The following images are taken from one such site. Unfortunately, I've lost track of that website.

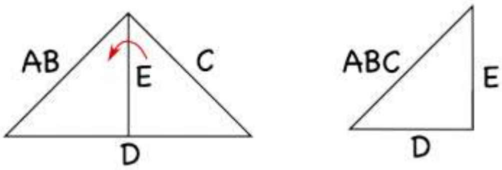
— Snowflakes —

This is a sequence of folds for creating 6-point snowflakes. Though it takes a few steps, don't be put off by them - with a little practice they become quick and easy.

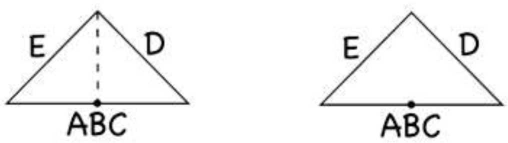
Start by taking a standard sheet of paper and folding at one of the corners so the sides marked A and B meet. Leave the fold in place and cut along the line marked C.



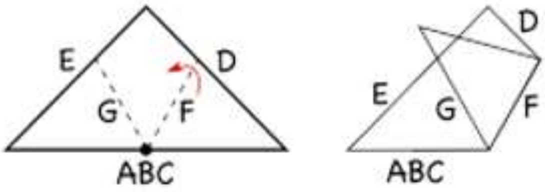
Take the triangle produced and fold it in half so that sides AB and C overlap.



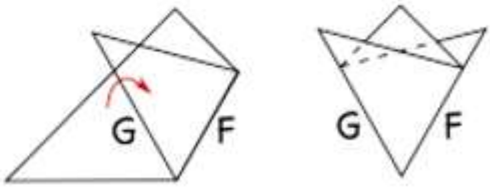
Put a temporary fold in this triangle and use the fold to mark the middle of side ABC. Undo the temporary fold.



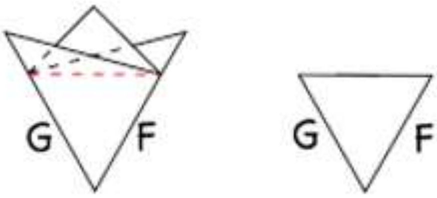
Make a fold over F. When you fold over F, you will be looking to have G placed so that G breaks the angle in half.



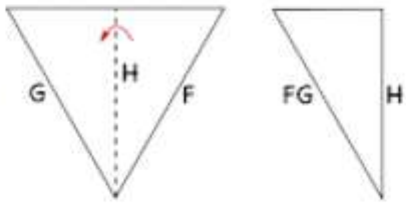
Fold along G - do this fold underneath so that this new folded piece is underneath the other paper.

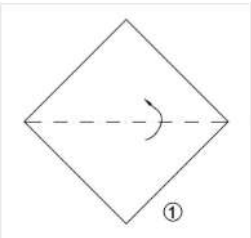


Although not strictly necessary, it is a good idea to slice off the top of this figure. Otherwise, you might be tempted to use the region above the red dotted line without realizing that there isn't paper on all levels.

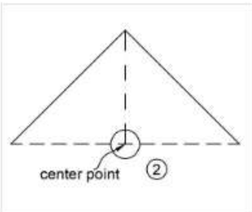


Lastly, fold this triangle in half. At last you are all set to do the cutting of your design!

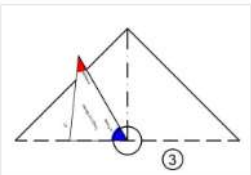




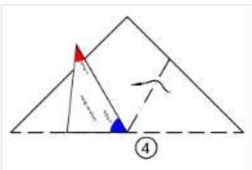
1. Follow the line fold the paper into half.



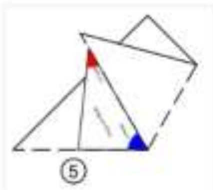
2. Find the center point at the folded side and make a crease.



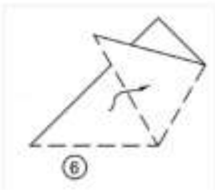
3. Put the template 6 points (blue corner) to the center point, as the picture 3 shows.



4. Fold the paper to the side of the template.

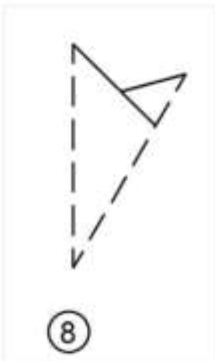


5. Then move away the template.

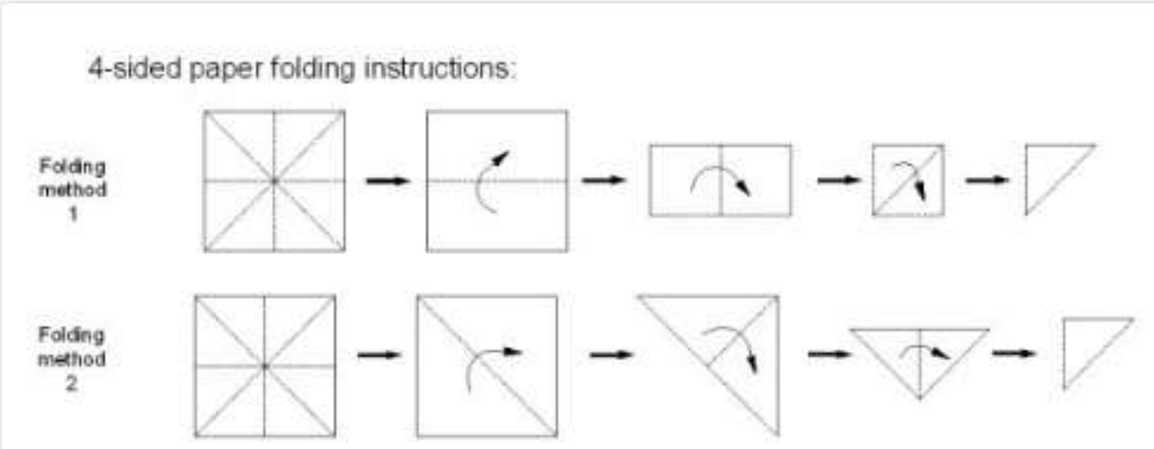


6. Fold the paper as the picture 6 showing.

7. Fold the paper again.



8. You finished a perfect 6 points or 6 repeats folding.



Kirigami Paper Cutting

Kirigami is a style of origami, but it involves cutting the paper as well as folding it. The term *kirigami* comes from the Japanese words *kiri*, which means "cut," and *kami*, which means "paper." This project uses a few simple shapes to make three-dimensional pop-outs.

Math in Action: *concentric circles, geometry, parallel lines, perpendicular lines*

What You'll Need

paper
cutting templates (see here)
scissors
thread or string

1 Copy one of the templates onto a piece of paper.

2 Fold the shape vertically through the center, where all the lines begin and end. This creates a center dividing line. Unfold, then fold the paper horizontally, matching up the lines on the left side.

3 Cut along the lines on the left side. Unfold. Refold horizontally, matching up the lines on the right side this time. Cut along the lines on the right side.

4 Unfold. Fold each section in alternating directions forward and backward, as shown. Hang by a thread as a decoration.

(Tripp: *Math Art and Drawing Games for Kids*)

In the book Claudia Zaslavsky - *Math games and activities from around the world* you can find many symmetrical patterns from the traditions of many communities.

Mon-Kiri Cutouts from Japan

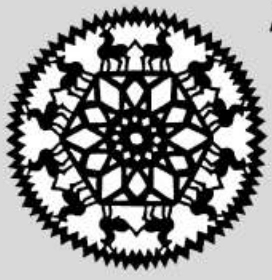


Figure 59a

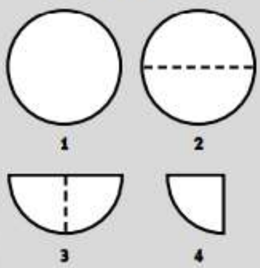


Figure 59b

The Chinese and Japanese people have a long history of cutting complicated designs in paper. In fact, archaeologists discovered a Chinese papercut that was made about 1,500 years ago. **Figure 59a**

In the old days warriors would decorate their armor with these designs. Now the designs appear as decorations in books, on people's homes, and in many other places. Paper cutouts are often used in making cartoon movies.

In Japan this art is called *mon-kiri*, the art of folding and cutting paper to make designs. You can make a beautiful cutout in the style of mon-kiri. You might want to give it to a family member or a friend for a birthday or valentine.

MATERIALS

- **5-inch (12.5cm) circle of white paper**
- **Pencil**
- **Scissors**
- **6-inch (15cm) circle of black construction paper**
- **Glue**
- **8-inch (20cm) square of red construction paper**

MAKING A MON-KIRI CUTOUT

1. Fold the white circle in half, then in half again. **Figure 59b**
2. Along the folded edges draw several designs that will be easy to cut out.
3. If you like, draw a border pattern along the rim of the circle.
4. Cut out the design and the border pattern. **Figure 59c**
5. Glue the white circle to the black circle so that the black border is even all around.
6. Glue the black circle to the red square so that the margins are even all around. **Figure 59d**

THINGS TO THINK ABOUT AND DO

Look at the mon-kiri you made. Imagine that you can draw a line and fold it so that one half fits over the other half. Can you draw two different fold lines? These are lines of symmetry.

Place a finger on the center of the mon-kiri. Give the circle a half turn. Now it looks the same as when you started. The pattern looks the same in two different positions.

Fold another white circle as directed above. Then fold it again, so that you have eight layers of paper. Draw and cut out designs. Before you open the circle, can you imagine how many times each design will appear?

Experiment with squares and circles of paper folded in different ways. Draw and cut out the designs. Before you open the paper, try to imagine how it will look. Then open the paper. Did you imagine it correctly?

Look at the patterns you made. How many lines of symmetry can you find in each pattern? In how many different positions does the pattern look the same when you turn it about the center?

Another form of Japanese papercutting is called *origami*. Read the book by Eleanor Coerr, *Sadako and the Thousand Paper Cranes*, a true story about a little Japanese girl who was poisoned by radiation when an atom bomb was dropped on her city during World War II. She and her friends believed that if they folded a thousand paper cranes, she would become well. Sadly, she died before they had finished the task.

Figure 59c

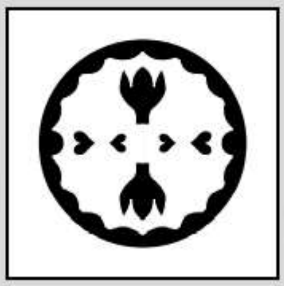


Figure 59d

(Claudia Zaslavsky - *Math games and activities from around the world*)

A44: Tesselations

Be A Tessellation Artist Islamic

You can learn to draw tessellations using only a compass and a straightedge, like the Islamic artists. This tessellation is made of equilateral triangles and regular hexagons. **Figure 68a**

MATERIALS

- **Large sheets of paper**
- **Compass**
- **Pencil**
- **Ruler or straightedge**
- **Colored markers or crayons**

DRAWING A TESSELLATION

You will start by constructing a regular hexagon using a compass and a straightedge. Then you will draw the six equilateral triangles to form a six-pointed star. With this star as the center, you can repeat the design in all directions.

1. Follow the directions on page 136 to construct a hexagon. Draw the construction lines lightly in pencil. **Figure 68b**
2. Extend the lines that form the sides of the hexagon. Now you have a six-pointed star. **Figure 68c**
3. Erase the circle, the hexagon, and the construction marks. Outline the star with a heavy pencil line. **Figure 68d**

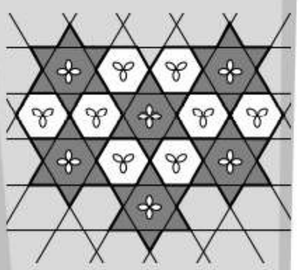


Figure 68a

Figure 68b

REPEATING PATTERNS

138

4. Draw six hexagons around the star. Here are directions to draw one hexagon. Use the same opening (radius) of the compass as in Step number 1.
   a. Place the point of the compass on the tip of the star at point A, and draw an arc at point C.
   b. Place the point of the compass on the tip of the star at point B, and draw an arc at point D.
   c. Then draw arcs from points C and D that intersect at point E. Draw a line through points C and E. Draw another line through points D and E. **Figure 68e**
   d. You now have a hexagon. Draw a heavy pencil line to outline it.
   e. Follow the directions above to draw five more hexagons around the star. You will find that you have already drawn some of the lines you need. You have also constructed many more triangles.
5. Continue as far as you can. Color the stars in one color and the hexagons in another color. Islamic artists liked to draw flowers in the center of each shape.

Don't be discouraged if your pattern is a bit lopsided. This craft takes a lot of patience. Keep at it, and you will have a beautiful work of art!

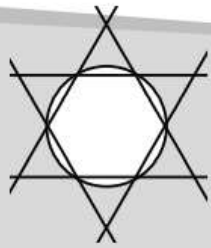


Figure 68c

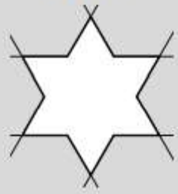


Figure 68d

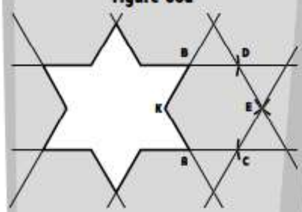


Figure 68e

REPEATING PATTERNS

139

(Claudia Zaslavsky - *Math games and activities from around the world*)

A45: Quilts

For the squares, or patches, women and girls used leftover bits of fabric and pieces of worn-out clothing. Nothing was wasted. Only a very special quilt or one made for a wealthy home rated large amounts of new material.

Let's design a simple quilt with paper squares. You should cut out at least thirty-six squares.

MATERIALS

- **Pencil**
- **Ruler**
- **2 or 3 sheets of white or light-colored paper**
- **Marker or crayon**
- **Scissors**

MAKING A PAPER QUILT

1. Use the pencil and ruler to divide each sheet of paper into two-inch (5cm) squares. You should have at least thirty-six squares. **Figure 64a**
2. Draw the diagonals of all the squares. Color each half of each square contrasting colors. **Figure 64b**
3. Cut out the squares.
4. Arrange the squares in a pleasing pattern to make a large quilt. Here are some ways to combine the squares. You can rotate some squares a quarter turn or a half turn. **Figure 64c**
5. You may want to keep the best pattern. Copy it on a sheet of construction paper or glue the squares to the sheet of paper. Draw a border around it as a frame, or glue it to a larger sheet of a contrasting color.

You have just created a *tessellation*. A tessellation is a pattern that covers a surface completely by combining certain shapes. Your tessellation is made up of squares.

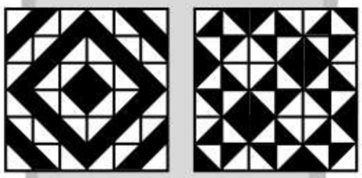


Figure 64c

Figure 64d

Figure 64d

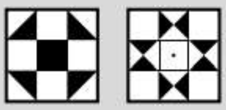


Figure 64e

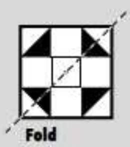


Figure 64f

THINGS TO DO

You can make interesting three-by-three quilt blocks. Each block will have nine squares. You need two sheets of paper in two different colors. Cut out nine squares of one color and nine squares of another color. Take four squares of each color and cut each square into two triangles. **Figure 64d**

Combine the squares and triangles to make a three-by-three quilt block. These are two traditional quilt blocks. They are called "Shoo Fly" and "Ohio Star." **Figure 64e**

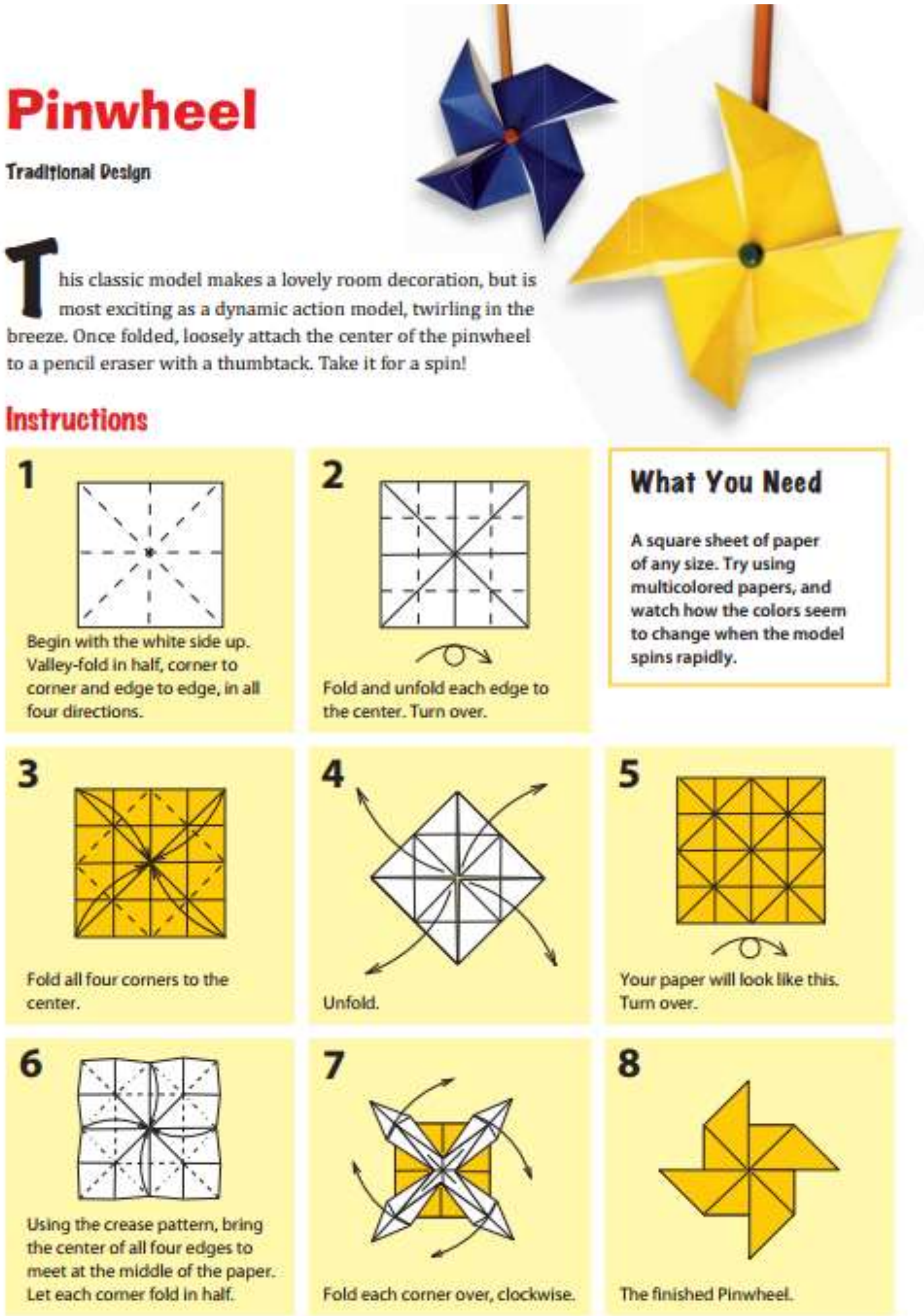
Both have symmetry. Look at each quilt block. Find four different turn positions. Then find four different fold lines. (See page 116 for a discussion of these terms.) Here is one fold line, a diagonal. **Figure 64f**

Sew a patchwork quilt cover out of cloth. First plan it on paper. When you cut the fabric, allow an extra centimeter all around for the seams.

Read books about quilts. Some describe different kinds of quilts in various cultures and how to make them. *The Quilt-Block History of Pioneer Days, with Projects Kids Can Make*, written for children by Mary Cobb, gives instructions in quilting. There are also children's storybooks that describe the importance of quilts in the lives of the characters. One example is Deborah Hopkinson's *Sweet Clara and the Freedom Quilt*.

(Claudia Zaslavsky - *Math games and activities from around the world*)

A46: Figures with paper and scissors. Various shapes can be obtained by bending and cutting paper. A lot of such material can be found on the Internet. They are especially famous **origami** https://en.wikipedia.org/wiki/Origami

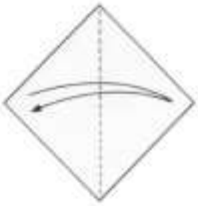


(Michael G. LaFosse - *Origami Activities for Kids*)

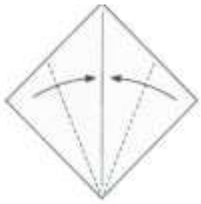
The following material is taken from the book Florence Temko: *Origami for beginners*

Kite Base, Pine Tree, Trick Mouse, Fantastic Flyer Airplane

KITE BASE (Use any square)



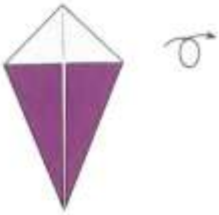
1. Fold in half. Unfold.



2. Fold in two edges to center crease.

3. Kite Base.

FANTASTIC FLYER AIRPLANE (Use any square and a paper clip)



1. Fold a Kite Base. Turn back to front.

2. Fold edges in.

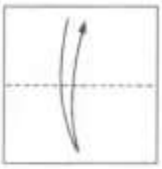
3. Mountain fold in half (refer to "Practical information", p. 3).

4. Place wings at 90° to the body. Attach a paper clip to the nose.

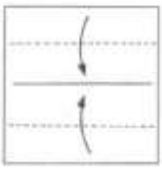
5. Fantastic Flyer Airplane. Throw airplane upward.

Boat, Sailboat Basket

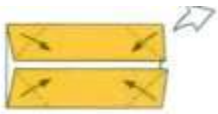
BOAT (Use any square)



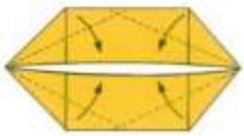
1. Fold in half. Unfold.



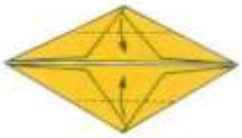
2. Fold edges in.



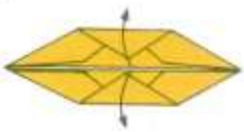
3. Fold four corners in.



4. Fold four corners to middle.



5. Fold corners to middle.



6. Reach in deep and spread sides apart.

7. Boat. Make a stronger model by returning to Step 1 and starting with the colored side up. Proceed to Step 7.

8. Pull top edges to the outside while pushing up bottom of boat, turning it inside out.

9. Inside-out Boat.

Bathtub Frolic

These paper boats float, but they last longer if they are made from waxed paper. To strengthen them even more, staple their sides.

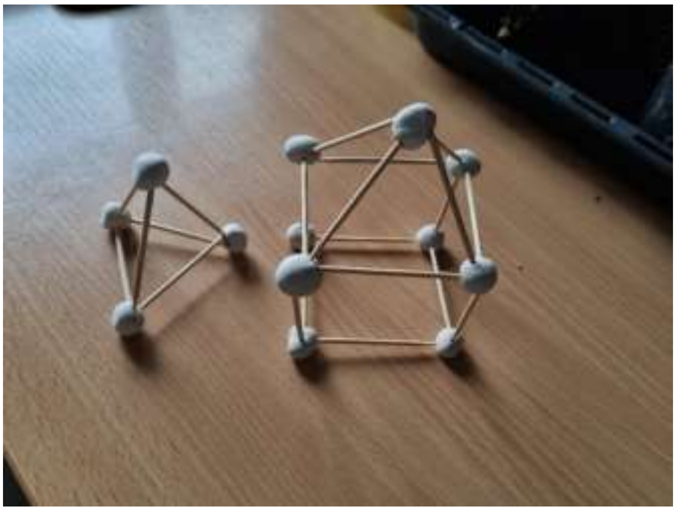
On the Internet, you can find a lot of good videos about making origami.

A47: Space figures

Regular space figures made of paper:

(https://www.minieco.co.uk/i-mathematics-platonic-solids-garland/)

Figures made of toothpicks and clay (or plasticine)



A48: Perspective. It is interesting to imagine what a spatial situation looks like when viewed from a certain position

Activity 2

With several photographs of the same model or scene from different perspectives *(e.g. front, sides and back)*, ask children to identify which of the photographs shows what the person can see? This could be one person or several people where the photographed perspectives could be matched to each person. The models can be recreated so that children can move around to experience the perspectives for themselves to check their answers.

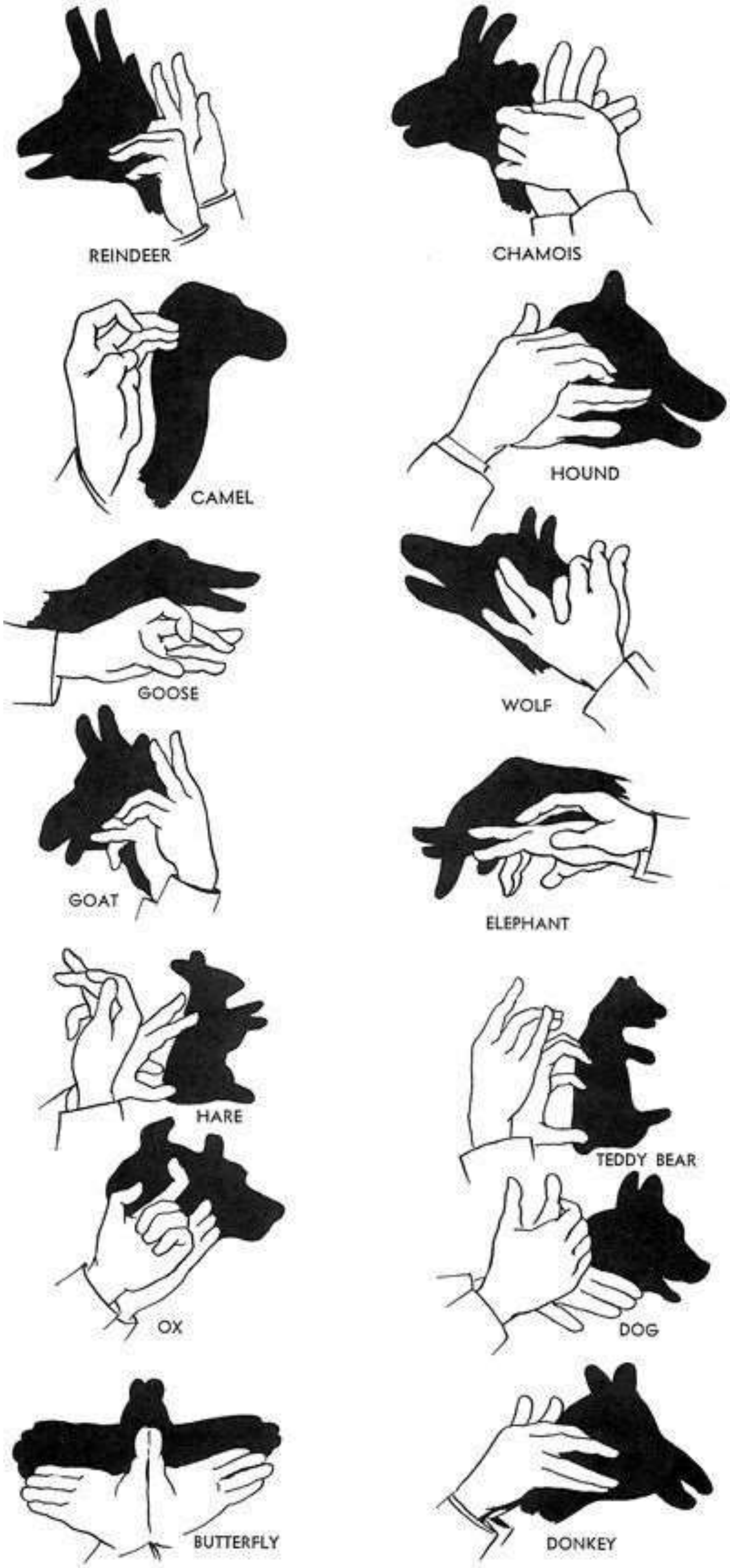
Which photograph did the child take? How do you know? Why can't it be photo 3? Frick et al. (2014)

Activity 3

Using a painting of a landscape or city from an art gallery's online collection, children can draw what can be seen from a specific perspective in the painting, perhaps from a specific place or through the eyes of a specific person depicted in the painting. This can lead to discussion about what can and can't be seen from that vantage point as well as how landmarks might appear to be a different shape from this perspective. Children might enjoy swapping pictures to work out the perspective that another child has captured in their drawing or they might be able to represent this alternative perspective using blocks, playdough or construction. Switching between two and three dimensional representations challenges thinking so drawing on whiteboards and representing in dough can be good to begin with as these can be adapted easily as children develop their ideas.

(the source is unknown to me)

A48: Projections. Making shadows with your fingers is a fun activity that also teaches us about projections. On the Internet, you can find many pages and videos that have this topic.



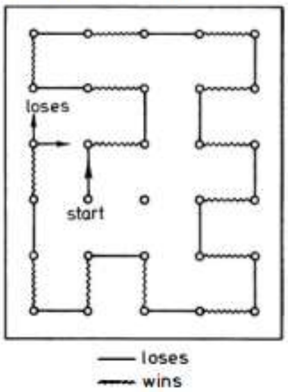
(https://www.pinterest.com/pin/227291730974234154/)

A49: Closed line game.

Snake

A Game for Two

This game is played on a five-by-six board of dots, like this one. Players take turns at joining two dots by a line to make one long snake. No diagonal lines are allowed. You cannot leave any breaks in the snake. Each player adds to the snake at either end; a player can only add to his opponent's segment, not to his own. The first to make the snake *close* on itself loses. Here is an actual game. In it straight lines began and lost.



(the source is unknown to me)

A50: Transforming by compressing and stretching (without cracking)

Materials

- ✔ Scissors
- ✔ Large rubber balloon
- ✔ Marker
- ✔ A plastic sandwich bag loosely filled with clay or playdoh and then closed
- ✔ Paper and pencil
- ✔ A ball, a small bowl or box, a mug with a handle, and a bagel (or any doughnut-shaped item with a hole in the middle)

MATH MEET

Name That Shape

Topologists *classify* shapes by putting them into categories. This is a fun contest for you and some friends. In five minutes:

- Who can classify the most shapes according to their number of holes?
- Who can find at least one object with zero, one, two, three, four, and five holes?
- Who can find the object with the most holes?
- Can you find a shape that your friends won't be able to classify? Bring your confusing shapes together and see if you can figure out how many holes they have!

In topology, you can stretch, squeeze, or twist a shape without changing what kind of shape it is. We'll explore how shapes are allowed to change in topology by transforming one shape into some others.

ACTIVITY 1: TRANSFORMING A CIRCLE

1. Cut the balloon in half lengthwise so that you have a sheet of rubber.

2. With the marker, draw a circle on the rubber sheet **(fig. 1)**.

3. Try to transform the circle into a square by pulling on the edges of the rubber sheet. You might need more than two hands to do it **(fig. 2)**!

4. Can you turn the circle into a triangle by pulling on the sheet? What other shapes can you transform the circle into? Because you didn't poke a hole, cut the sheet, tape parts of it together, or draw another line, topologists consider all of these the same shape.

FIG. 1: *Draw a circle onto the rubber sheet.*

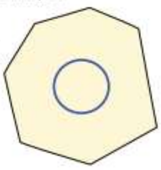
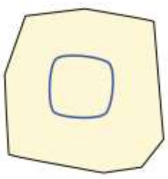


FIG. 2: *Pull on the edges of the rubber sheet to transform the circle into a square and other shapes.*



(Rapaport: *Math Lab for Kids*)

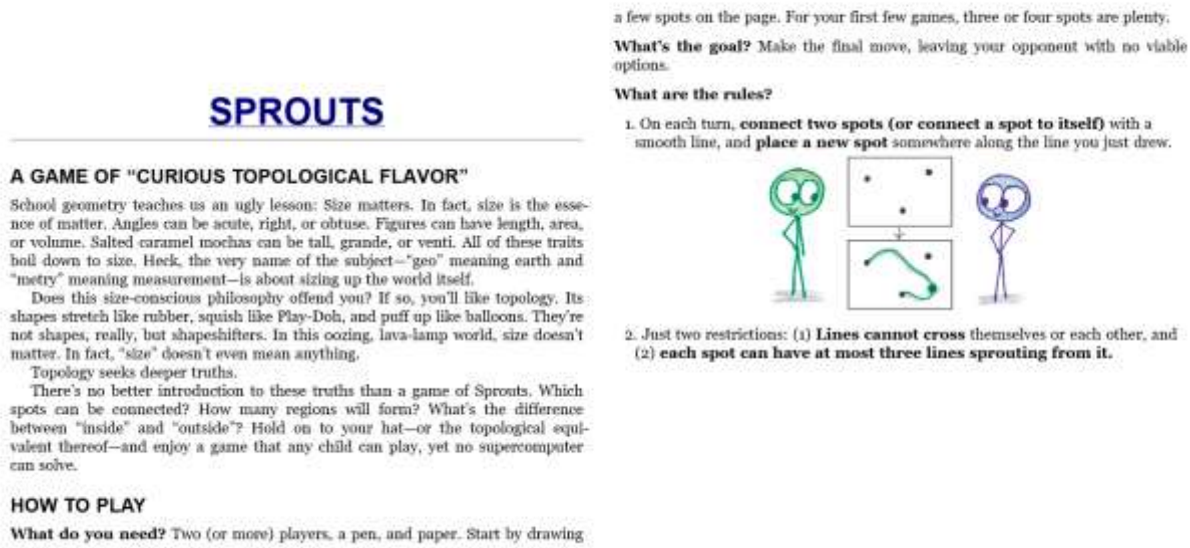
A51: Dressing.

Order of dressing (what must be put on before what).

How to put a jacket on the floor so that when you put your hands in the sleeves, you get dressed properly.

Puzzle. You enter one hole, you exit through two. When you went out, you only then came in (pants).

A52: Sprouts – connecting dots game.



SPROUTS

A GAME OF "CURIOUS TOPOLOGICAL FLAVOR"

School geometry teaches us an ugly lesson: Size matters. In fact, size is the essence of matter. Angles can be acute, right, or obtuse. Figures can have length, area, or volume. Salted caramel mochas can be tall, grande, or venti. All of these traits boil down to size. Heck, the very name of the subject—"geo" meaning earth and "metry" meaning measurement—is about sizing up the world itself.

Does this size-conscious philosophy offend you? If so, you'll like topology. Its shapes stretch like rubber, squish like Play-Doh, and puff up like balloons. They're not shapes, really, but shapeshifters. In this oozing, lava-lamp world, size doesn't matter. In fact, "size" doesn't even mean anything.

Topology seeks deeper truths.

There's no better introduction to these truths than a game of Sprouts. Which spots can be connected? How many regions will form? What's the difference between "inside" and "outside"? Hold on to your hat—or the topological equivalent thereof—and enjoy a game that any child can play, yet no supercomputer can solve.

HOW TO PLAY

What do you need? Two (or more) players, a pen, and paper. Start by drawing a few spots on the page. For your first few games, three or four spots are plenty.

What's the goal? Make the final move, leaving your opponent with no viable options.

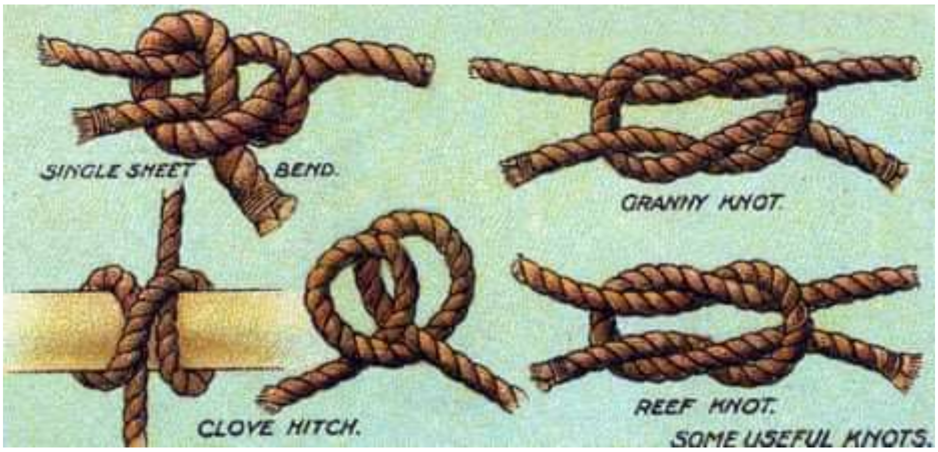
What are the rules?

1. On each turn, **connect two spots (or connect a spot to itself)** with a smooth line, and **place a new spot** somewhere along the line you just drew.

2. Just two restrictions: (1) **Lines cannot cross** themselves or each other, and (2) **each spot can have at most three lines sprouting from it.**

(Orlin: *Math Games with Bad Drawings*)

A53: Useful nodes. Tying shoelaces is a good activity. On the Internet you can find many videos and sites showing how to make knots for various purposes. At the following link there is a brochure that explains how to make basic scout knots: https://www.scoutadventures.org.uk/sites/default/files/2018-05/Simple%20Knots%20-%20Factsheet_0.pdf



(https://www.theguardian.com/lifeandstyle/shortcuts/2014/nov/03/joy-knots-scouts-reef-tie)

A54: Puzzles with knots. You can find videos on the Internet with various „tricks“ with knots. One trick is also in the story in Appendix 1: the ends of the rope are caught and a knot must be made on the rope without letting go of the ends.

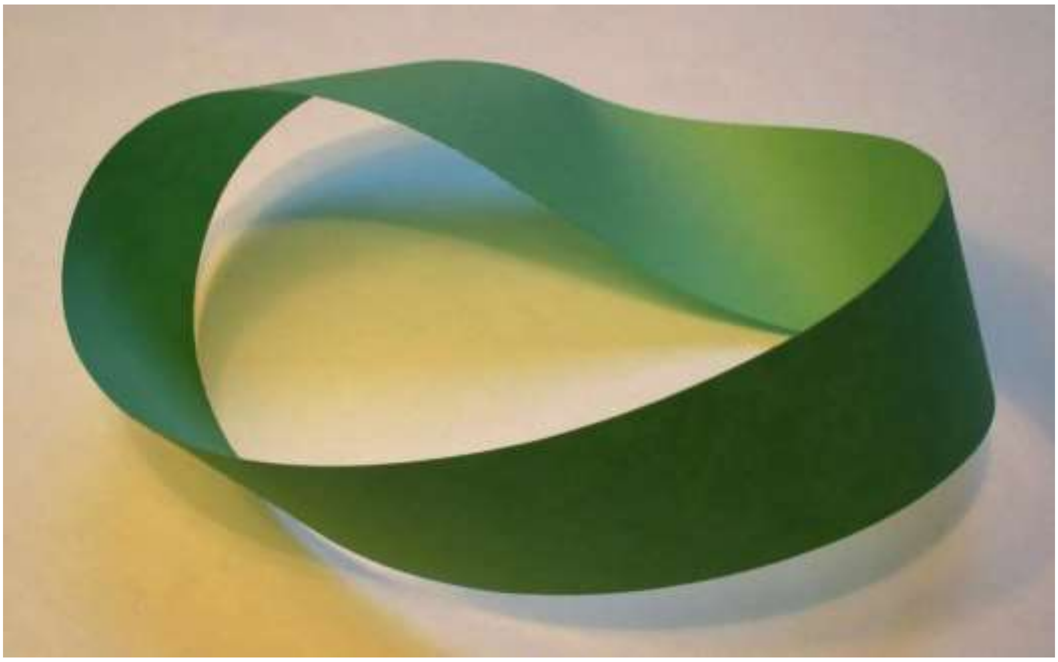
A55 Knitting.

(Dejić: Matematika kao igra)

There are many videos and websites on the Internet about knitting for children. This picture was taken from one such place.

https://www.howwemontessori.com/how-we-montessori/2018/09/montessori-inspired-finger-knitting.html

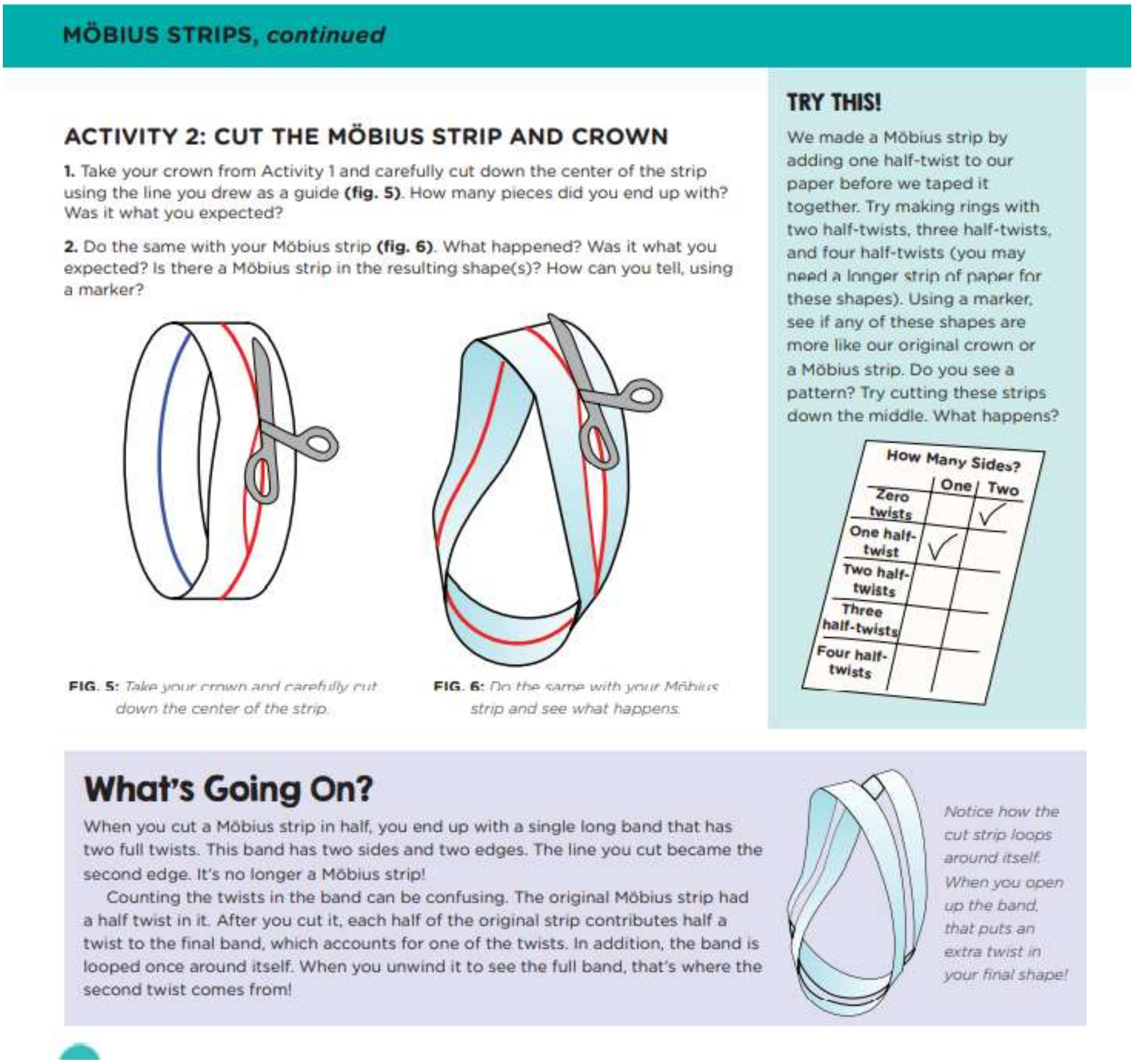
A56 : Möbius strip. We get it by twisting the tape before joining it in a circle.



https://en.wikipedia.org/wiki/M%C3%B6bius_strip

We can do „miracles" with the Möbius strip. Here is an introductory puzzle. We have a strip of paper, let's say a meter long, scissors and scotch tape. You need to glue the paper and then cut it, where nothing else should be glued or unglued, and get a circle with a circumference of 2 meters (two meters long). Solution: We need to connect the strip into a Möbius strip and cut it lengthwise, as shown below

:



MÖBIUS STRIPS, *continued*

ACTIVITY 2: CUT THE MÖBIUS STRIP AND CROWN

1. Take your crown from Activity 1 and carefully cut down the center of the strip using the line you drew as a guide **(fig. 5)**. How many pieces did you end up with? Was it what you expected?

2. Do the same with your Möbius strip **(fig. 6)**. What happened? Was it what you expected? Is there a Möbius strip in the resulting shape(s)? How can you tell, using a marker?

FIG. 5: *Take your crown and carefully cut down the center of the strip.*

FIG. 6: *Do the same with your Möbius strip and see what happens.*

TRY THIS!

We made a Möbius strip by adding one half-twist to our paper before we taped it together. Try making rings with two half-twists, three half-twists, and four half-twists (you may need a longer strip of paper for these shapes). Using a marker, see if any of these shapes are more like our original crown or a Möbius strip. Do you see a pattern? Try cutting these strips down the middle. What happens?

How Many Sides?

	One	Two
Zero twists		✓
One half-twist	✓	
Two half-twists		
Three half-twists		
Four half-twists		

What's Going On?

When you cut a Möbius strip in half, you end up with a single long band that has two full twists. This band has two sides and two edges. The line you cut became the second edge. It's no longer a Möbius strip!

Counting the twists in the band can be confusing. The original Möbius strip had a half twist in it. After you cut it, each half of the original strip contributes half a twist to the final band, which accounts for one of the twists. In addition, the band is looped once around itself. When you unwind it to see the full band, that's where the second twist comes from!

Notice how the cut strip loops around itself. When you open up the band, that puts an extra twist in your final shape!

(from the book Rapoport: *Math Lab for Kids*.)

Below are some more suggestions from the mentioned book:

Materials

- ✔ Two strips of white paper about 2 inches (5 cm) wide and 22 to 24 inches (56 to 61 cm) long
  You can make these by taping two strips cut from an 8.5 x 11-inch (21.6 x 27.9 cm) sheet together, but make sure the tape covers the whole width of the strip.
- ✔ Tape
- ✔ Markers in at least two different colors
- ✔ Scissors

ACTIVITY 3: CUT A MÖBIUS STRIP AND CROWN INTO THIRDS

1. Make a new paper crown and a new Möbius strip with the strips of paper.

2. Carefully draw a line around the crown again, this time about a third of the way from the edge. (Don't worry if it's not exactly a third.) Using a different color, draw another line about a third of the way from the other edge **(fig. 1)**. Do the same thing with your Möbius strip **(fig. 2)**. What is different between the lines you drew on the crown and on the Möbius strip?

3. Cut the paper crown along the lines that you drew **(fig. 3)**. What shapes do you end up with?

4. Before you cut your Möbius strip into thirds, try to guess what shape or shapes you'll end up with—how many pieces, with how many twists? Once you've made your guess, cut the Möbius strip using your lines as a guide **(fig. 4)**. What do you end up with? Is it what you expected? Use a marker to figure out if any of the shapes are Möbius strips.

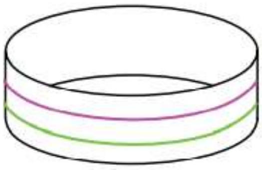


FIG. 1: *Carefully draw a line around the crown about a third of the way from each edge.*

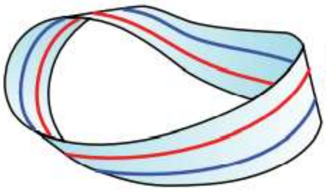


FIG. 2: *Do the same thing with your Möbius strip.*

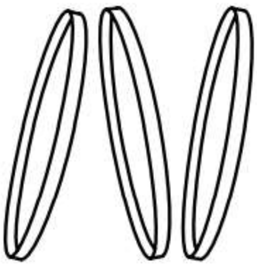


FIG. 3: *Cut the paper crown along the lines that you drew.*

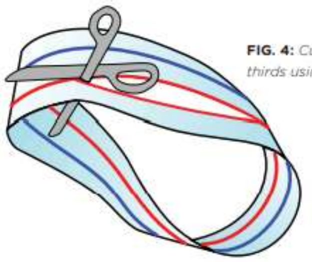


FIG. 4: *Cut the Möbius strip into thirds using your lines as a guide.*

LAB 11 MÖBIUS SURPRISE

Materials

- ✔ Paper (8.5 x 11 inches [21.6 x 27.9 cm])
- ✔ Markers in two different colors
- ✔ Tape
- ✔ Scissors

Martin Gardner, a mathematician famous for introducing fun mathematical challenges to the public, invented an entertaining surprise using the concepts we just learned. Try it for yourself!

TRY THE MÖBIUS SURPRISE

1. Draw a thick plus sign on a piece of white paper. Cut out the shape. Draw a single solid line across the short arm of the plus sign. Turn the shape over, and draw the same line on the back. Draw two dotted vertical lines dividing the long arm of the plus sign into equal widths. Turn the shape over and repeat these lines on the back of the paper **(fig. 1)**.

2. Take the two horizontal arms, with the single solid line on them, and tape the edges together without any twists to form a ring. Make sure the tape goes all the way across the joint so that it won't fall apart later **(fig. 2)**.

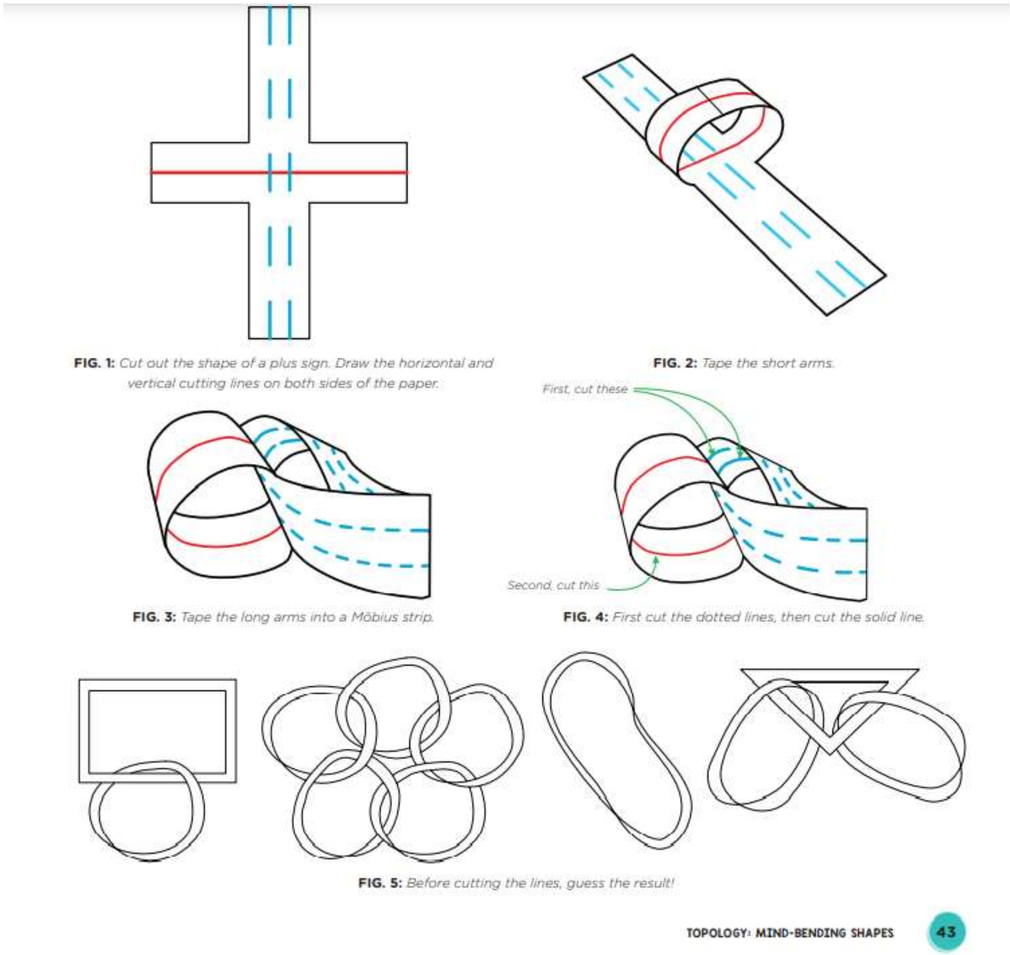
3. Take the remaining two arms and tape them into a Möbius strip opposite your original ring **(fig. 3)**.

4. Before you cut along your lines, try to guess what the final shape will be **(fig. 5)**. Will it be a giant ring? Several interlocked rings? Some other shape?

5. The order that you cut the lines is important. First, cut along the dotted lines (they should be on the twisted ring of the surprise). Next, cut along the solid line **(fig. 4)**. What do you end up with?

TRY THIS!

The Möbius surprise is made by connecting a ring and a Möbius strip and cutting them. Try inventing other combinations of shapes and twists, cutting them, and seeing what you end up with. Can you invent a surprise shape named after you?



A57: Navigation by instructions. We give the children instructions on where the hidden object is - orally or on paper if the children can read.

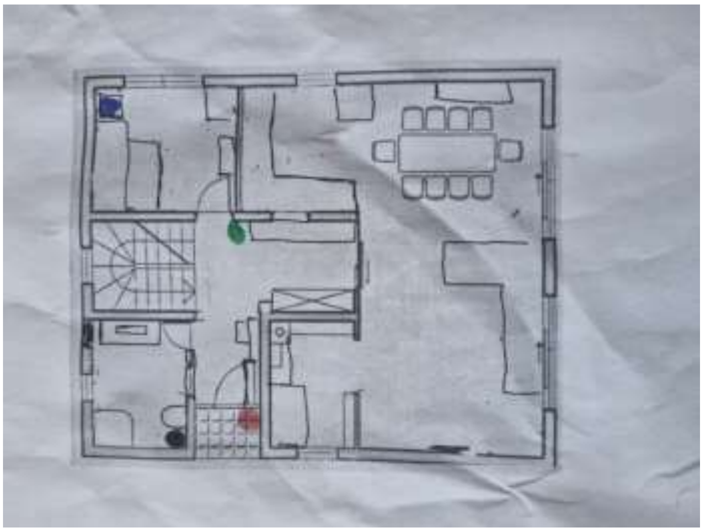
(the source is unknown to me)

A58: Map navigation. Map of a room, apartment, neighborhood, entire place, globe, and getting children used to being able to follow on maps how to get from one place to another.

A59: **3D maps**. They can be found here and there, for example a 3D map of the city center or a mountain. They are very interesting for children: they can, for example, recognize familiar places in the city and see how to get to those places. Here is a 3D map of Zagreb:

(https://profitiraj.hr/otkrivena-skulptura-grada-zagreba/)

A60: Searching for a marked item on the map. This can be done in nature. An object is hidden, the children are given a map by which they can find it. Or in the apartment. For example, an object is placed somewhere in the apartment and then the child searches for it using the map. When she learns to read the map, she can also put the object and mark on the map where she put it, so that the other child can look for it.



(Some places were marked where Darko Ban hid gifts for Nina)

A61: Sketching the path and movement to a place using straight and curved arrows. For example marking the way to school. Or a walk around the zoo in Zagreb:

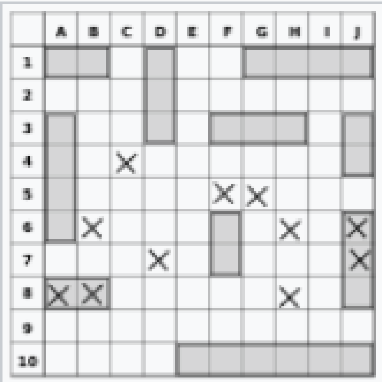
A62: View from above. How the landscape looks to a bird flying over an area. How a bird sees our house, neighborhood, etc. *Google Earth* can also help here:

A63: Making maps.

(the source is unknown to me)

A64: Trip planning.

A65: Coordinate system - Battleship Game. https://en.wikipedia.org/wiki/Battleship_(game) Appropriate symbols can be used for coordinates, if the children have not yet mastered letters or numbers. It is played on ruled grids (paper or board) on which each player's fleet of warships are marked. The locations of the fleets are concealed from the other player. Players alternate turns calling "shots" at the other player's ships, and the objective of the game is to destroy the opposing player's fleet.

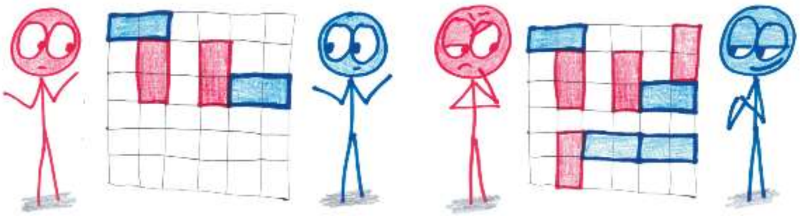


A map of one player's ships and the hits against them, from a game in progress. The grey boxes are the ships placed by the player, and the cross marks show the squares that their opponent has fired upon. The player would be tracking the success of their own shots in a separate grid

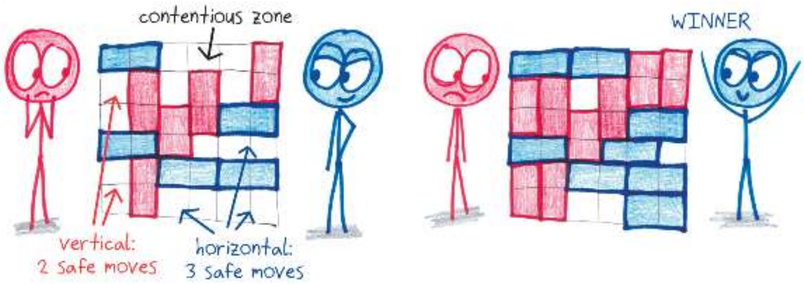
A66: Some games „in coordinates“.

A GAME OF CROWDING DOMINOS

In this game, two players take turns placing dominos on a rectangular grid. **One player lays dominos vertically; the other, horizontally.** (You can ignore the numbers on the dominos.) If it is your turn and you have **nowhere to place a domino, then you lose**.



The early moves feel a bit random. But soon, corridors start to appear. You and your opponent vie to secure “safe” spots for the future. Eventually, the board breaks down into disconnected chunks, and you can tally exactly how many moves each player has left.



(Orlin: *Math Games with Bad Drawings*)

Also known as Stop-Gate or Cross-Cram, the game is a classic of combinatorial game theory, prominently featured in the canonical text *Winning Ways for Your Mathematical Plays*. Whereas other classics (such as the million variations on Nim) serve better for mathematical analysis than casual gameplay, I find that Domineering works on both levels.

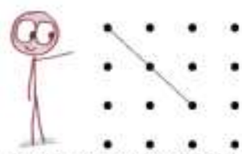
By the way, no need for actual dominos. You can **play by filling in squares on a paper grid.**

HOLD THAT LINE

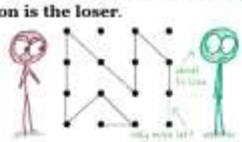
A GAME OF SNAKING GROWTH

Sid Sackson devised this game as an alternative to tic-tac-toe. “If all the people who ever played Tic-Tac-Toe were laid end to end,” he wrote, “they would promptly fall asleep.” He hoped to replace those drawish doldrums with a more flavorful game, one that never ends in a tie.

To begin, draw a 4-by-4 array of dots. The **first player connects any two dots with a straight line of any length**, which may be vertical, horizontal, or 45° diagonal.



Then, **take turns extending this line from either end** by drawing another line (vertical, horizontal, or 45° diagonal). Extensions may be any length, but may not cross or touch. Play until no further extensions are possible. The **person to draw the last extension is the loser**.

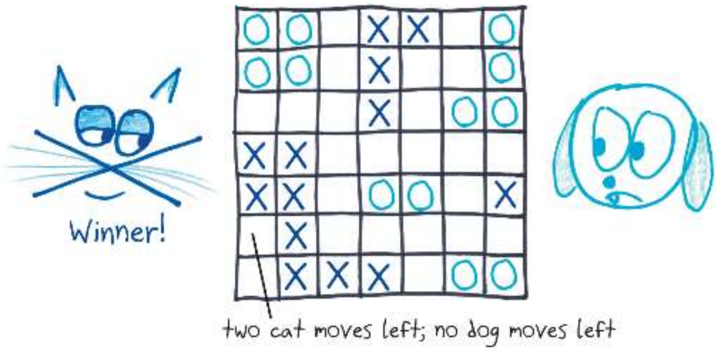


Sid's game resembles an older one published by Édouard Lucas alongside Dots and Boxes. Édouard's version has just a few differences: (1) You play on a 6-by-6 array of dots. (2) Each move must be a short vertical or horizontal line, connecting two adjacent dots. (3) You must build off of your opponent's most recent move, meaning that the “snake” grows from only one end. (4) The last player to move is the winner.

(Orlin: *Math Games with Bad Drawings*)

A GAME OF REFUSING TO GET ALONG

Draw a 7-by-7 grid on paper. Then, **take turns placing your respective animals**: for one player, "cats" (i.e., X's), and for the other, "dogs" (i.e., O's). **Cats and dogs must never occupy neighboring squares**, not even diagonally. **The last player able to move is the winner.**[8]

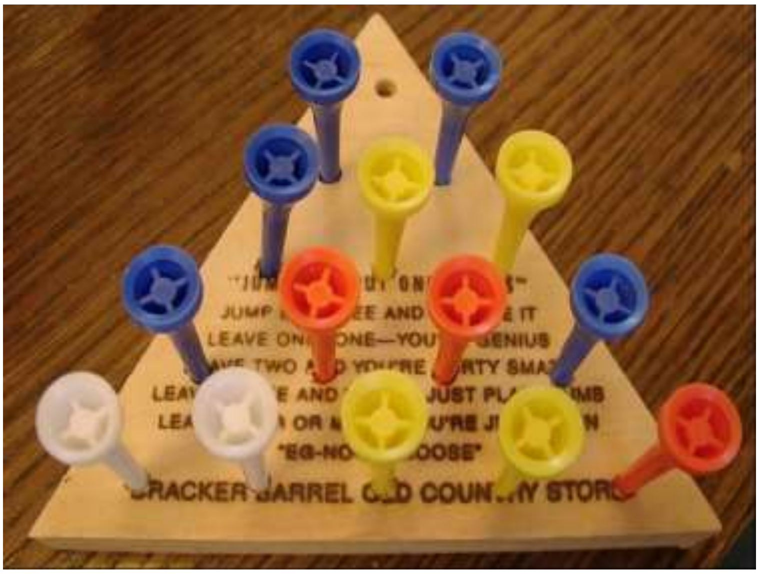


The game was developed by algebraist Simon Norton and was known in his honor as "Snort." Instead of cats and dogs, he imagined bulls and cows, grazing in various fields, liable to noisy snorts of distraction if placed too close to the opposite sex.[9] The fields need not follow a grid arrangement; you can draw any convoluted map of regions that you like.

A related classic of combinatorial game theory, Col, reverses the key rule: *Opposite*-species neighbors are allowed, and *same*-species neighbors are forbidden. Col is easier to analyze mathematically and, perhaps for that reason, less fun to play. Scattering your cats around the board, trying to keep them apart? Meh. Fencing off safe territories with jagged walls of cats? Now *that's* fun.

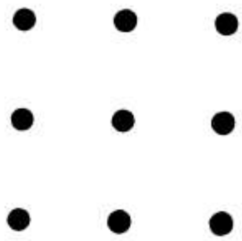
(Orlin: *Math Games with Bad Drawings*)

A67: Peg solitaire. You may remove pegs by jumping over them with another peg, as in checkers. Your goal is to jump these pegs over each other, one by one, until only a single peg is remaining (taken from https://www.gamesforyoungminds.com/blog/2018/4/13/peg-solitaire).

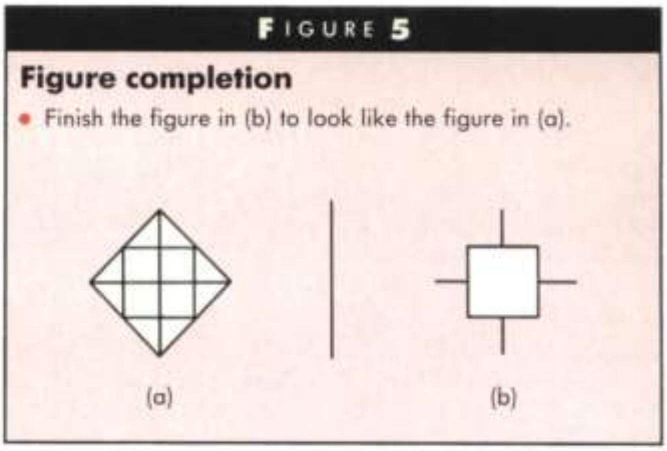


A68: Geometric puzzles.

Make a square with 9 dots as shown. Cross all the dots with 4 straight lines without taking your pencil off the paper.

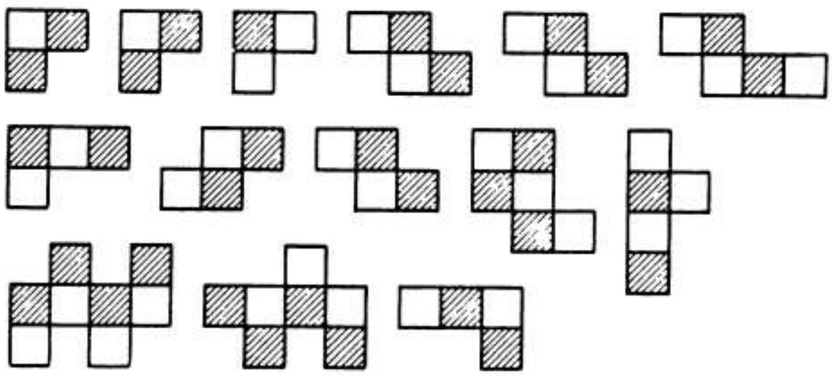


(the source is unknown to me)



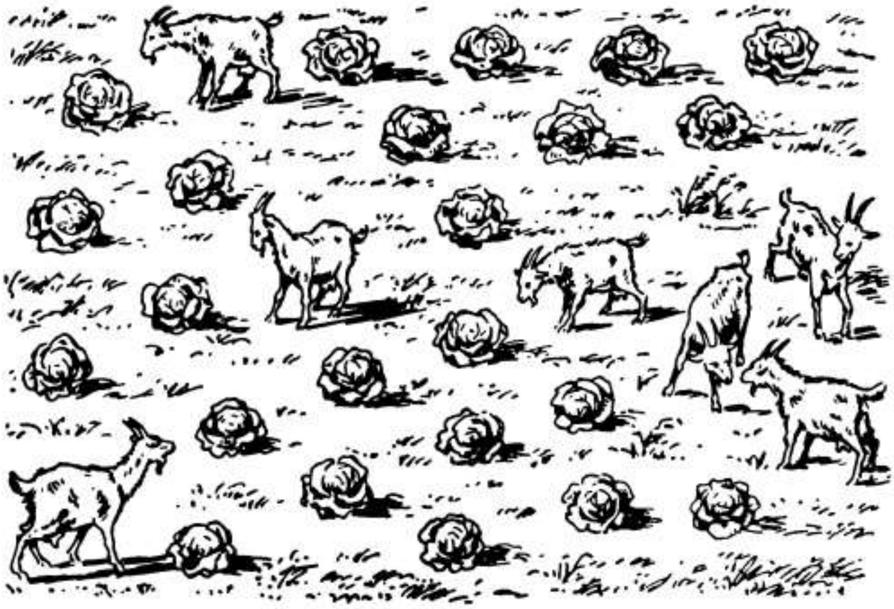
(the source is unknown to me)

A merry chess player cut his cardboard chessboard into 14 parts, as shown. Friends who wanted to play chess with him had to put the parts back together again first.



(from the book Kordemsky: *The Moscow Puzzles*)

The Moscow Puzzles



25. GOATS FROM CABBAGE

Now, instead of joining points, separate all the goats from the cabbage in the picture by drawing 3 straight lines.

(Kordemsky: *The Moscow Puzzles*)

7 Language

You can read about the great importance of language for the (mathematical) development of a child on the page 17.

A1: Telling stories to children.

A2: Encouraging the child to tell stories herself.

A3: Expressing feelings and describing situations. Encourage the child to verbally accompany what she is doing, what she is witnessing, what she is feeling.

A4: Playing with words. Making up unusual words. The child should describe what it is - what is the meaning of that word.

A5: Playing with speech sounds. Encourage the child to understand that spoken words are made up of speech sounds (phonemes) and sentences are made up of words. For example to try to change the order of speech sounds in a word, say the word from the end, or change the order of the words in the sentence.

A6: **Spoken and written language.** Encourage the child to connect the written text with the spoken text.

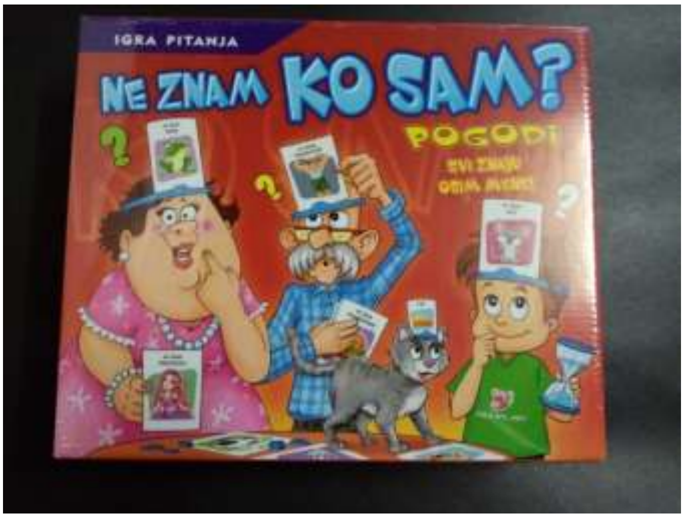
A7: Encryption. One simple method of encryption is explained on page 47.

8 Logic

You can briefly read about the distinct presence of logic in children's world on page 19.

A1. Logical and consistent thinking. It is most important for the development of logic, when children use language to describe and analyze a situation or when they imagine something, that they develop a logical sequence of thoughts, logically draw conclusions, and avoid inconsistencies. They should develop the same approach when they listen to what others tell them. It is precisely these activities that we should support and encourage.

A2: Guessing the hidden object (character). Game *Who am I, what am I?*



But commercial versions are not required for games of this type. It is enough for some to imagine an object (or hide it in a box) and others to guess.

Mystery Box

Materials

The only materials you will need are a box (a shoe box or similarly sized box will do) and any household object that will fit inside the box.

Procedure

Children love riddles and mysteries. This activity is a favorite because of a child's curiosity to find out what is in the box. Put the object in the box so that your child does not know what it is. He or she can only find out what the mystery object is by asking questions that can be answered by a yes or no.

At first, your child will try to guess the object outright. (For example: "Is it a ball?") Encourage him or her to be a good detective and ask questions that will provide clues. (For example: "Is it round?" or, "Is it red?") Clues for your child to consider are color, size, shape, use, and so on. As your child become mores proficient at this activity, you may want to have a limit on the number of questions asked, as in the game Twenty Questions.

(the source is unknown to me)

A3: Connectives. Selection of objects that meet some conditions combined with conjunctions *and, or, not*.

the house has blue curtains and there are flowers in front of it

Otok ima palmu, nema bijelu kuću i nema jezero.

(*Fora zbirka Super zadataka*)

— MAKE ME A LIAR —

ACTIVITY

Someone makes a statement and the other players attempt to show that the person is lying.

One type of statement is to say something is always true. Examples of this are: all trucks have four wheels, all rectangles are squares, all birds can fly, and the moon comes out at night.

Another type of statement is of the form "if __, then ___." Examples of this are: if today is Monday, then it is a school day; if I don't eat for three hours, then I am hungry; and if a person is taller than someone, then they are older.

(the source is unknown to me)

A4: Quintifiers *all* and *some*: The truth of statements of the type *All are A, All A are B, there is some A, some A are B, not all A are B, all A are not B, etc.* is tested. For example, we line up some toys - figures and say „All toys are animals“ or „Some toys are animals“ or „Every animal is dog or cat“, and the children determine whether it is true or false. Afterwards, they compose such sentences themselves and we answer.

A5: Testing the truth of complex sentences. In order to be able to examine the truth of more complex sentences in addition to simpler sentences, we must have objects with various properties and mutual relationships - a rich context. It can be a real situation, for example the gathering of children and parents at a party. So we can ask, for example, if someone has a child whose father or mother did not come, but whose older brother or sister is there, etc.

For setting up complex sentences, it is convenient to have the so-called **Dienes logical blocks**, figures that come in various shapes, colors, sizes and thicknesses. Below are some possibilities

ATTRIBUTES

	Shape	Size	Color	Thickness
VALUES	square rectangle triangle circle ellipse (oval)	small medium large	red blue yellow green	thin (skinny) thick (fat)

Such wooden blocks are, for example, WISSNER 39740 Active Learning Logical Blocks Expansion Set Attributes Blocks 60 Pieces - RE-Wood. I got them through Amazon.de for 15 euros plus postage.

Let's select one pile of such blocks. Then, from that pile, we select blocks that meet a certain condition. We can combine two values (blue triangle, blue or triangle, not blue and not triangle) or more (large green circle, large not green or circle).

Or we select a pile and examine the truth of some statements about that pile (all blue figures are triangles, some large figures are not blue, etc.)

In the story in Appendix 1, a problem related to creating logical conditions and solving them is given.

A6: Puzzles

Four cats line up for mittens: a white cat, a yellow cat, a brown cat, and black cat. The white cat is first in line. The black cat is in front of the yellow cat. The brown cat is in front of the black cat. What color is the last cat in line?

Billy’s mother had five children. The first was named Lala, the second was named Lele, the third was named Lili, the fourth was named Lolo. What was the fifth child named?

You’re driving a city bus. At the first stop, three women get on. At the second stop, one woman gets off and a man gets on. At the third stop, two children get on. The bus is blue and it’s raining outside in December. What colour is the bus driver’s hair?

Are there more uphills or downhills in Croatia? More left or right curves on the roads?

A man was walking in the rain in the middle of nowhere without a coat or an umbrella. He got soaked, but not a single hair on his head was wet. How can this be?

9 Procedural thinking

You can read briefly about procedural thinking in the children's world on page 19.

A1 recipes and instructions. For cooking (an example is given on page **Error! Bookmark not defined.**) or for performing any physical task. For example, instructions on how to go to the store, how to feed the cat, how to make a paper boat. Instructions on how to get to the hidden object.

A2 Labyrinths – There is a simple algorithm for how a child can find something in the maze, or find a way out of the maze, or return to the entrance from which she entered the maze. You can apply it to all maze tasks. It would be best to make a real maze in the yard. This algorithm is based on the old Greek story of the Minotaur. Ariadne gave the Greek hero Theseus a ball of rope (Ariad's thread) and instructions on how to search the labyrinth to find the Minotaur, and how to return when he found him. Until he finds the Minotaur, Theseus unrolls the ball and does the following. When he is at an intersection, he goes to a corridor in which there are no threads (it means that he was not there). If there are threads in all the corridors, then he returns through the corridor from which he first came. It is the only corridor in which there is one thread - in all the others there are two threads because he applied one when he entered that corridor and the other when he returned. Whether Theseus went down a certain corridor or came back down a certain corridor, he will again find himself at a crossroads where he applies this instruction again. This procedure ensures that they will pass all the corridors of the maze and return back. Of course, when Theseus finds the Minotaur, he will immediately go back through the corridors where there is only one thread.

If the goal is to enter the labyrinth through one entrance and exit through the other, and the labyrinth does not have roundabouts, then we have a simpler algorithm - we constantly follow the left (or right) wall of the labyrinth.

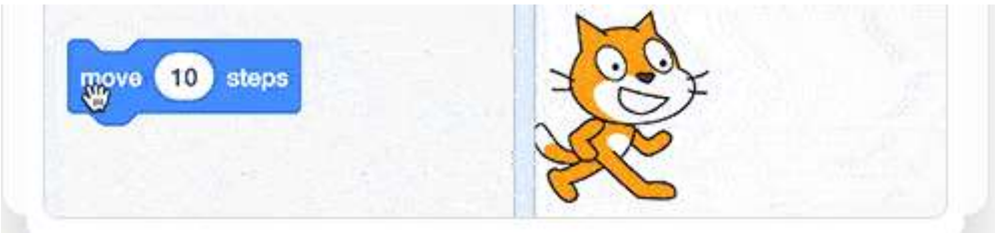
A3: ***The Enormous Turnip*** (https://en.wikipedia.org/wiki/The_Gigantic_Turnip) is a Slavic children's story that illustrates the programming idea of repeating an action until something comes true. Type the title of the story in the search field at https://archive.org/ and you will get multiple versions of the story.

(https://tesla.carnet.hr/mod/book/view.php?id=6681&chapterid=1440)

A4: A Hen is Going to the Fair is a South Slavic children's story that illustrates the programming idea of reducing one problem to another. One procedure calls another, this calls the third, and when the last one is executed, then all of these are terminated. The story can be found in Croatian[26] at the link https://more.rivrtici.hr/sites/default/files/posla_koka_u_ducan_0.pdf. There is also an animated film in Croatian that you can find at https://www.youtube.com/watch?v=fjIbqYPrPCQ

[26] This is a wonderful story with clear mathematical content and it is a pity that it is not freely available in English. I only know of the English translation in the book Stanić Rašin: *When Hen Was on Her Way to Market*, Perlina Press, 2018.

A5: Programing. Scratch is a programming language intended for children to learn programming in a visual way. It is very interesting and suitable for children, although parents should be careful when to start with it and how much time to allow the child to spend in front of the screen. Here is the text from their site https://scratch.mit.edu/ : Scratch is the world's largest coding community for children and a coding language with a simple visual interface that allows young people to create digital stories, games, and animations. Scratch is designed, developed, and moderated by the *Scratch Foundation*, a nonprofit organization.



10 Problem solving

On page 20 it is pointed out that problem solving is a natural environment for developing mathematics. In the previous topics, many activities involved problem solving either directly, or one would have to invent such a context in which these activities would naturally appear. Here are some examples.

A1 Crossing a river.

11. WOLF, GOAT, AND CABBAGE

This problem can be found in eighth-century writings.

A man has to take a wolf, a goat, and some cabbage across a river. His rowboat has enough room for the man plus either the wolf or the goat or the cabbage. If he takes the cabbage with him, the wolf will eat the goat. If he takes the wolf, the goat will eat the cabbage. Only when the man is present are the goat and the cabbage safe from their enemies. All the same, the man carries wolf, goat, and cabbage across the river.

How?

(Kordemsky: The Moscow Puzzles)

10. CROSSING A RIVER

A detachment of soldiers must cross a river. The bridge is broken, the river is deep. What to do? Suddenly the officer in charge spots 2 boys playing in a rowboat by the shore. The boat is so tiny, however, that it can only hold 2 boys or 1 soldier. Still, all the soldiers succeed in crossing the river in the boat. How?

Solve this problem either in your mind or practically—that is, by moving checkers, matches, or the like on a table across an imaginary river.

(Kordemsky: The Moscow Puzzles)

A2 Riddles.

It has scissors, it is not a tailor,

live in water, it is not a fish

wears armor, it is not a soldier. (cancer)

It has a mustache, it is not a grandpa,
drinks milk, it is not a child. (cat)

Graze the grass,
it's not a pig.
It has horns, no legs.
And without legs he walks through the field,
and without a home it will not go anywhere.
(snail)

Everyone who loves it

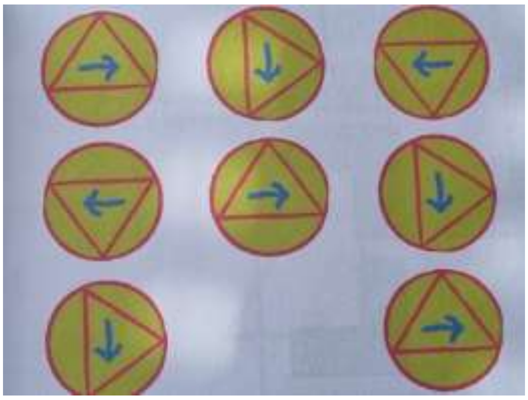
they beat it

they kick it

and they throw it out of their hands.

(Ball)

A3 What is missing?



(Dejić: Matematika kroz igru)

3 Appendix

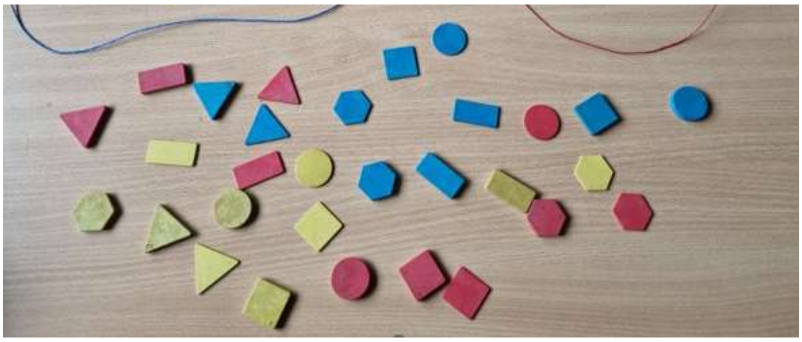
1 CHILDREN, BEES AND DWARFS

©Boris

This is an example of how mathematical content can be incorporated into a story in such a way that children are naturally motivated to develop appropriate mathematics in the context of the story. Below are three versions of the story, mathematically speaking the medium, the easier and the harder version. Depending on the current children's needs, or the needs of a group of children, different versions can be made. If this story is read to children, then the appropriate material must be provided, and the mathematical content must be demonstrated in cooperation and discussion with the children. Even better, the story can serve as the basis for a performance in which the children themselves will perform the mathematical content as actors, and children from the audience can also join in.

MEDIUM VERSION

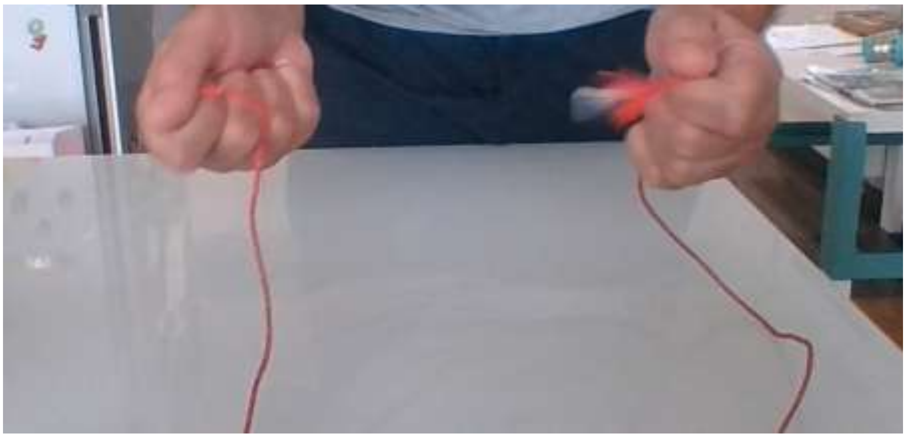
Materials needed: three ropes about a meter long, several paper strips about a meter long with scotch tape and scissors, and the figures shown in the following picture:



*

Spring has come. Nina, Ezra and Tara happily ran to the meadows above the houses. Their play was interrupted by a cry they heard nearby, at the very edge of the forest. When they ran there, they had something to see: the whole bee colony was crying! „Why are you crying?“ the children asked them. The bees answered that they were crying because when the people mowed the meadows they also cut down the dandelions. Without dandelions, they cannot prepare food for the hive and their babies will starve. „Can we help you in any way?“ asked the worried children. The bees answered that there are still dandelions only in one meadow, to which the path leads through a strait guarded by three dwarfs and which do not allow anyone to pass. They only let go of whoever gives them a puzzle they can't solve. So far they have solved all the puzzles and have not let anyone go. „We'll come up with puzzles for them that they won't be able to solve“, the children were determined. They sat down in the meadow and thought, thought, thought, ... - and came up with it. They quickly ran to the house and took ropes, paper strips, duct tape, scissors and various wooden shapes.

First they encountered the dwarf Vilko. Small, with a red cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have“, answered the children: „Here, grab the ends of this rope with your hands and make a knot without letting go of the ends.“. „Ha, nothing easier than that.“ replied Vilko confidently. He caught the ends of the rope:



He wound the rope, twirled his hands, twisted the rope, even uttered the magic words "abrakadabrakaodjutradomraka", and the like:

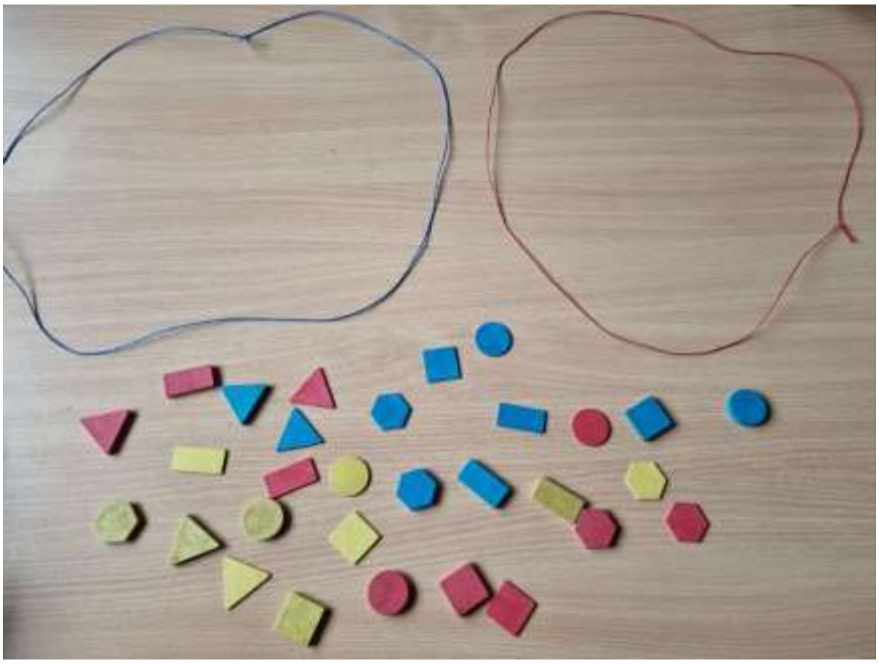
But always when he unwound the rope there would be no knot:

In the end, he got terribly angry and started shouting: „That's impossible! You can't make a knot like that!“. „It is possible“, the children calmly answered him, „Now we will show you“. Ezra crossed his arms first and then took the rope.

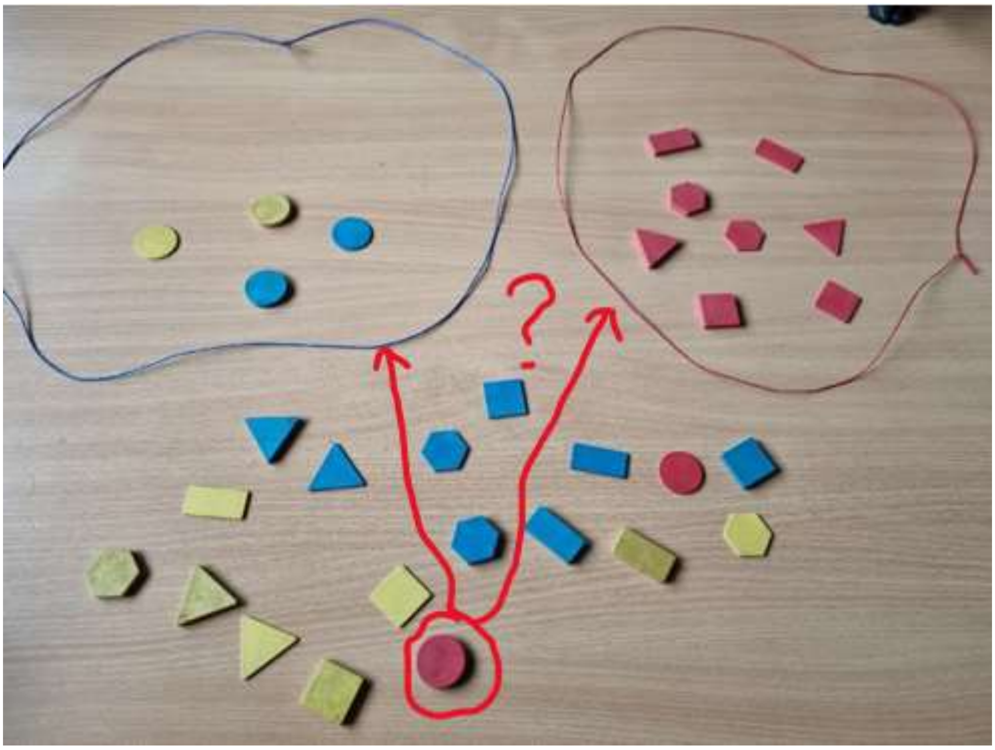
When he spread his arms, the rope knotted:

When he saw that it could be done, that he just had to take the ends of the rope differently, Vilko became even angrier. Out of anger, he started jumping in circles and screaming. When he calmed down a bit, he admitted that the task could still be solved and let the children and the bees pass. But he also threatened: „You passed me, but you certainly won't pass my older brother Milko. He is soooo smart and will surely solve your puzzle.“. This worried the bees, but the children comforted them: „Don't worry bees, we have an even more difficult puzzle.“.

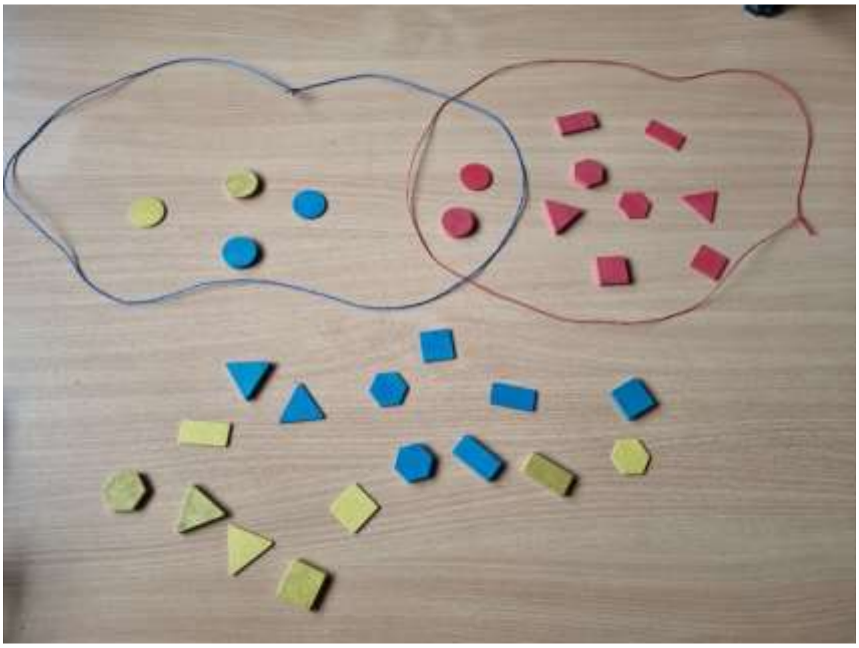
Soon they came across the dwarf Milko. Small, with a blue cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have“, answered the children: „There are various figures and two ropes here.



You have to put the round figures in one fence, and the red figures in another.“ „Ah, that's easy.“ said Milko, and began to put the round figures in one fence and the red figures in the other. But when he took the red circle, he didn't know where to put it, because he would have to put it in both fences:



„That is impossible. This task cannot be solved.“ concluded Milko confidently. „Now you will see that it is possible.“ said the children. Tara placed the fences so that the red circles could be placed in both fences:



When he saw that the puzzle had a solution after all, the dwarf Milko started to blush, blush and blush, ... Steam started coming out of his ears, and his eyes rolled. The children worried that he was not well, gave him some water to drink and wiped his forehead with a damp cloth. When he regained his composure, Milko tearfully screamed: „You're right. Go through. In any case, you will not pass the eldest brother Pilko. He is the smaaartest dwarf and will surely solve your puzzle“. This worried the bees, but the children comforted them: „Don't worry, bees, we have a particularly difficult puzzle for him.“

And so, they arrived in front of the third dwarf Pilko. Small, with a yellow cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have,“ answered the children: „Here you have a paper strip, scotch tape and scissors.“

„First, use scotch tape to connect the parts of the strip. Then cut the strip with scissors so that you get two circles that are connected to each other like links in a chain.“, the children gave him instructions. „Ha, now I'll solve it right away“, Pilko started confidently. He glued the ends of the tape and cut the tape lengthwise with scissors:

But instead of connected ones, he got separated circles:

„Wait, now I'll solve it", Pilko fidgeted. He glued and cut the strips several times, but he never managed to get two connected circles. He began to fidget, to sweat, and finally to jump angrily and shout: „That's impossible. This task cannot be solved. It's too complicated. Oh, my head hurts. Do you have any medicine for headache?“. The bees gave him some propolis, which stopped his headache, and Nina showed him how to get rid of it. She took one end of the tape,

she turned the tape full circle,

taped it to the other end of the tape,

cut the strip lengthwise with scissors ,

and got two connected circles.

While Pilko looked at the solution in disbelief, the children and the bees reached a beautiful meadow full of yellow dandelions. The bees filled their bags and on their way back, the whole forest echoed with their happy song. When it was time to say goodbye, they asked the children how to thank them. The children answered that their greatest gratitude was to see them happily buzzing with nature. Then they hurried home, so that their parents wouldn't worry, because it was already getting dark.

That year, the orchards were full of fruit. Parents could not be surprised at how many there were. They said that there had never been so much fruit in any year. The children just smiled, because they knew the answer.

EASIER VERSION

Spring has come. Nina, Ezra and Tara happily ran to the meadows above the houses. Their play was interrupted by a cry they heard nearby, at the very edge of the forest. When they ran there, they had something to see: the whole bee colony was crying! „Why are you crying?“ the children asked them. The bees answered that they were crying because when the people mowed the meadows they also cut down the dandelions. Without dandelions, they cannot prepare food for the hive and their babies will starve. „Can we help you in any way?“ asked the worried children. The bees answered that there are still dandelions only in one meadow, to which the path leads through a strait guarded by three dwarfs and which do not allow anyone to pass. They only let go of whoever gives them a riddle they can't solve. So far they have solved all the riddles and have not let anyone go. „We'll come up with riddles for them that they won't be able to solve“, the children were determined. They sat down in the meadow and thought, thought, thought, ... - and came up with it. They quickly ran to the house and took ropes, paper strips, duct tape, scissors and various wooden shapes.

First they encountered the dwarf Vilko. Small, with a red cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any riddle for me?“. „We have“, answered the children: „Tell us, are there more uphills or downhills in Zagreb?“. „What kind of riddle is that? How could I know how many uphills and downhills there are in Zagreb? That can't be solved without going around the whole city,“ Vilko was angry. „It can“, said the children, „And it's very simple.“ If when you go on one side it is uphill, then when you go on the other side it is downhill. Therefore, in Zagreb there are equal uphills and downhills.“[27] When he heard the solution, Vilko became even angrier. Out of anger, he started jumping in circles and screaming. When he calmed down a bit, he admitted that the riddle could still be solved and let the children and the bees pass. But he also threatened: „You passed me, but you certainly won't pass my older brother Milko. He is soooo smart and will surely solve your riddle.“. This worried the bees, but the children comforted them: „Don't worry bees, we have an even more difficult puzzle.“.

Soon they came across the dwarf Milko. Small, with a blue cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any riddle for me?“. „We have“, answered the children: „Tell us, are there more left or right curves on the roads in Croatia?“. „What kind of puzzle is that“, Milko was angry, „I would have to go around the whole of Croatia and examine all the curves on roads in order to solve it. That cannot be solved by thinking.“ - Milko was determined. „Yes“ said the children, „If when you go in one direction it is a right curve, then when you go in the opposite direction it is a left curve. That's why there are equal left and right curves in Croatia“.[28] When he saw that the riddle had a solution after all, the dwarf Milko started to blush, blush and blush, ... Steam started

[27] This can be easily illustrated by climbing with a sled in the snow and descending with a sled down the snow. Or, if retelling live, with a slanted piece of paper.

[28] This can be easily illustrated with a two-way road and cars going in two directions. For one driver, the curve is on the left, and for the other, on the right. Or, if the story is being told live, we fold a piece of paper, and the child touching it with the closest hand goes around in one direction, so when she comes back she has to change the hand she is touching the paper with.

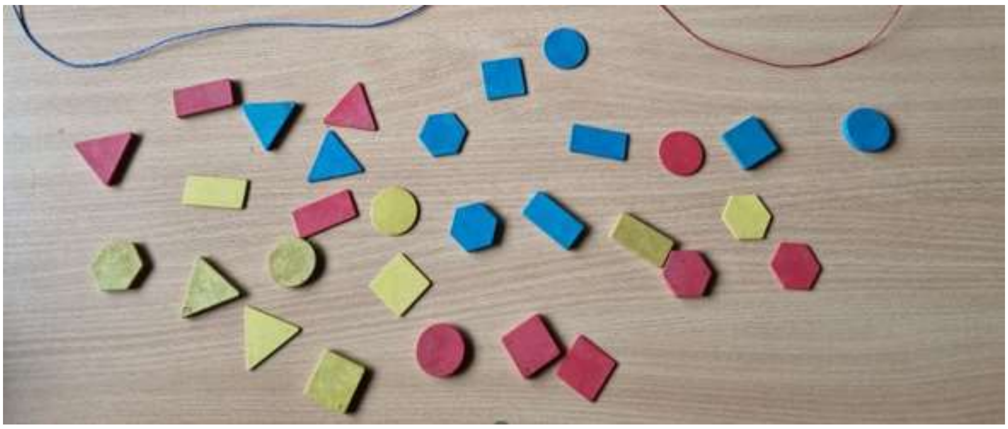
coming out of his ears, and his eyes rolled. The children worried that he was not well, gave him some water to drink and wiped his forehead with a damp cloth. When he regained his composure, Milko tearfully screamed: „You're right. Go through. In any case, you will not pass the eldest brother Pilko. He is the smartest dwarf and will surely solve your riddle“. This worried the bees, but the children comforted them: „Don't worry, bees, we have a particularly difficult puzzle for him.“

And so, they arrived in front of the third dwarf Pilko. Small, with a yellow cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any riddle for me?“. „We have,“ answered the children: „You go in on one hole, you go out on two. When you came out, you only then came in. What is that?“. „What kind of riddle is that?“, Pilko was angry, „How can someone get out of two holes? And come in only when he came out? That's nonsense!“, Pilko started shouting angrily. Poor Pilko, no matter how hard he tried to solve the puzzle, he failed. Finally, he began to jump angrily and shout: „That's impossible. This task cannot be solved. It's too complicated. Oh, my head hurts. Do you have any medicine for headache?“. The bees gave him some propolis, which stopped his headache. The children then explained to him that the solution was: pants. „Ha, now I understand“ said Pilko. He was very sad that he didn't know how to solve it, so the children and the bees had to comfort him. When they had comforted him, they went to a beautiful meadow full of yellow dandelions. The bees filled their bags and on their way back, the whole forest echoed with their happy song. When it was time to say goodbye, they asked the children how to thank them. The children answered that their greatest gratitude was to see them happily buzzing with nature. Then they hurried home, so that their parents wouldn't worry, because it was already getting dark.

That year, the orchards were full of fruit. Parents could not be surprised at how many there were. They said that there had never been so much fruit in any year. The children just smiled, because they knew the answer.

HARDER VERSION

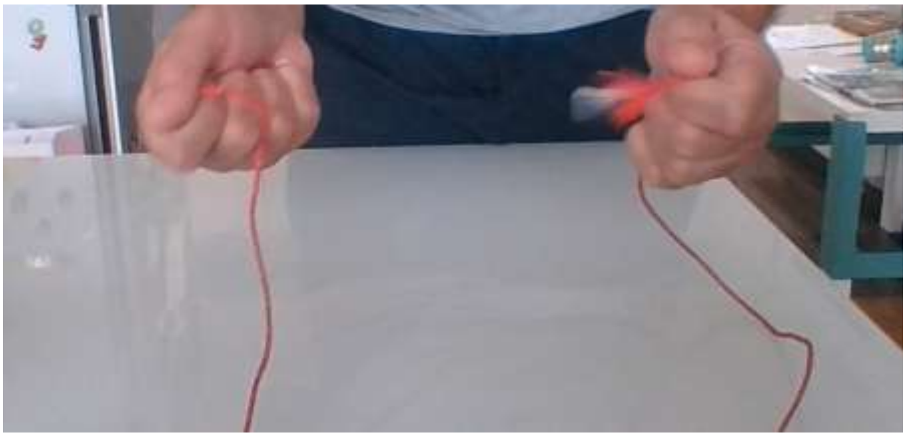
Compared to the *Medium version*, only the last puzzle is different.
Required material: three ropes about a meter long, and the figures shown in the following picture.



*

Spring has come. Nina, Ezra and Tara happily ran to the meadows above the houses. Their play was interrupted by a cry they heard nearby, at the very edge of the forest. When they ran there, they had something to see: the whole bee colony was crying! „Why are you crying?“ the children asked them. The bees answered that they were crying because when the people mowed the meadows they also cut down the dandelions. Without dandelions, they cannot prepare food for the hive and their babies will starve. „Can we help you in any way?“ asked the worried children. The bees answered that there are still dandelions only in one meadow, to which the path leads through a strait guarded by three dwarfs and which do not allow anyone to pass. They only let go of whoever gives them a puzzle they can't solve. So far they have solved all the puzzles and have not let anyone go. „We'll come up with puzzles for them that they won't be able to solve“, the children were determined. They sat down in the meadow and thought, thought, thought, ... - and came up with it. They quickly ran to the house and took ropes, and various wooden shapes.

First they encountered the dwarf Vilko. Small, with a red cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have“, answered the children: „Here, grab the ends of this rope with your hands and make a knot without letting go of the ends.“. „Ha, nothing easier than that.“ replied Vilko confidently. He caught the ends of the rope:



He wound the rope, twirled his hands, twisted the rope, even uttered the magic words "abrakadabrakaodjutradomraka", and the like:

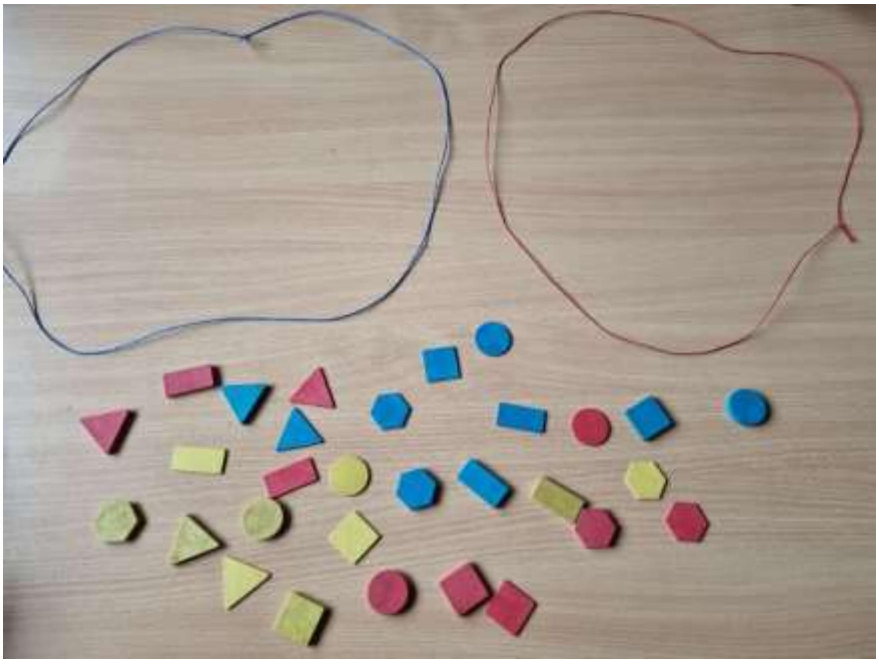
But always when he unwound the rope there would be no knot:

In the end, he got terribly angry and started shouting: „That's impossible! You can't make a knot like that!“. „It is possible“, the children calmly answered him, „Now we will show you“. Ezra crossed his arms first and then took the rope.

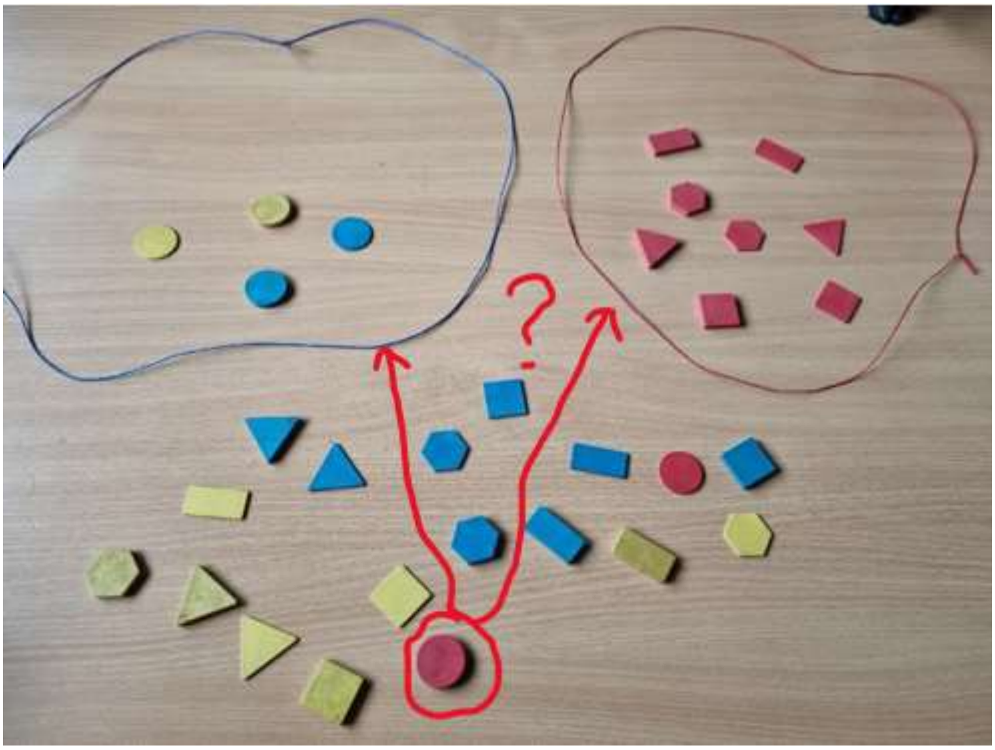
When he spread his arms, the rope knotted:

When he saw that it could be done, that he just had to take the ends of the rope differently, Vilko became even angrier. Out of anger, he started jumping in circles and screaming. When he calmed down a bit, he admitted that the task could still be solved and let the children and the bees pass. But he also threatened: „You passed me, but you certainly won't pass my older brother Milko. He is soooo smart and will surely solve your puzzle.“. This worried the bees, but the children comforted them: „Don't worry bees, we have an even more difficult puzzle.“.

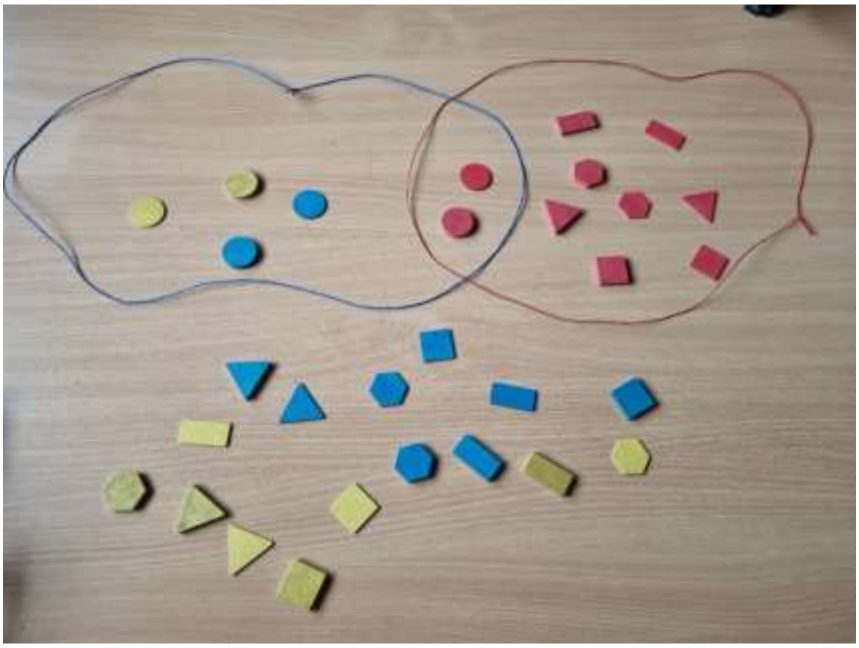
Soon they came across the dwarf Milko. Small, with a blue cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have“, answered the children: „There are various figures and two ropes here.



You have to put the round figures in one fence, and the red figures in another.“ „Ah, that's easy.“ said Milko, and began to put the round figures in one fence and the red figures in the other. But when he took the red circle, he didn't know where to put it, because he would have to put it in both fences:



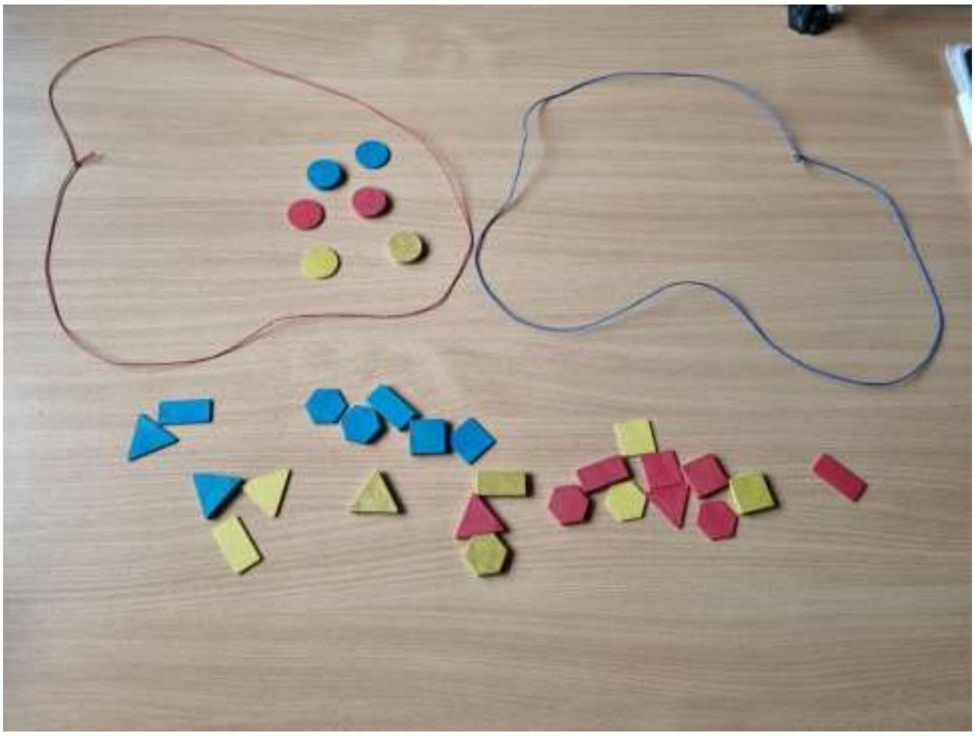
„That is impossible. This task cannot be solved.“ concluded Milko confidently. „Now you will see that it is possible.“ said the children. Tara placed the fences so that the red circles could be placed in both fences:



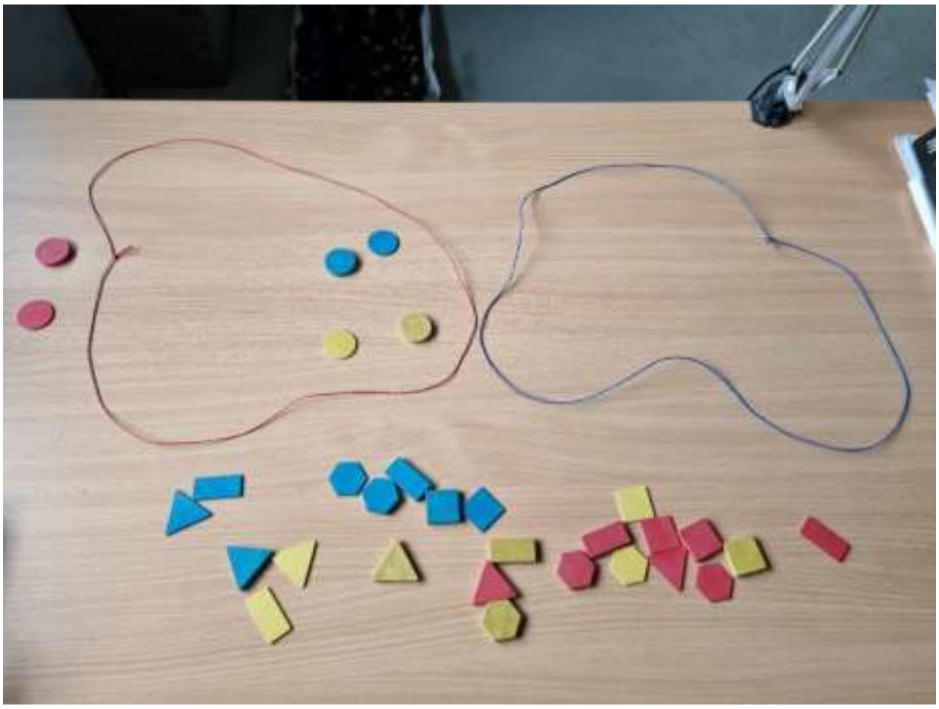
When he saw that the puzzle had a solution after all, the dwarf Milko started to blush, blush and blush, ... Steam started coming out of his ears, and his eyes rolled. The children worried that he was not well, gave him some water to drink and wiped his forehead with a damp cloth. When he regained his composure, Milko tearfully screamed: „You're right. Go through. In any case, you will not pass the eldest brother Pilko. He is the smartest dwarf and will surely solve your Puzzle“. This worried the bees, but the children comforted them: „Don't worry, bees, we have a particularly difficult puzzle for him.“

And so, they arrived in front of the third dwarf Pilko. Small, with a yellow cap, pointy beard and pointy boots, smilingly rubbed his hands and asked them: „Do you have any puzzle for me?“. „We have“, answered the children: „Here you have various figure and two ropes. You need to fence off all the figures that are round and not red or blue and not square.“. „Ha, I'll solve it right away“, started Pilko briskly, but already for the first figure he couldn't decide whether that figure was round and not...: „How did you say that, children?“, Pilko was confused. „Round and not red or blue and not square“, the children repeat. „I'll do it now“ Pilko encouraged himself, but he couldn't remember what kind of figures he needed to separate. „Round and not red or blue and not square“ - the children had to repeat to him constantly. Poor Pilko, no matter how hard he tried and tried to pick out the „round and... How is it going?“, in the end he started jumping angrily and shouting: „That's impossible. This task cannot be solved. It's too complicated. Oh, my head hurts. Do you have any medicine for headache?“. The bees gave him some propolis, which stopped his headache. Nina showed him how to solve it. She explained to him slowly, step by step, so that he wouldn't get a headache again.

„Pilko, when you have a difficult task, you first have to break it down into parts that are easier to solve,“ explained Nina, „Instead of looking for all the figures that are round and not red or blue and not square, which gives you a headache, first we will find all figures that are round and not red. And in order to find them, we will first find all the round figures. And that's easy, look:"

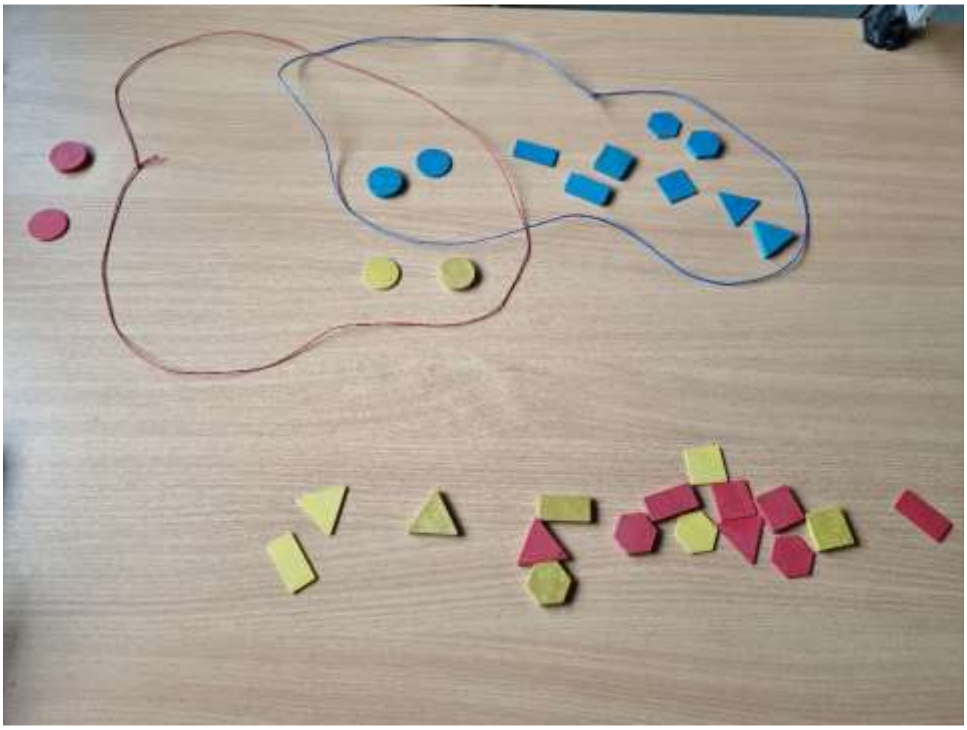


„Now we will take among them those who are not red. And that's easy: we'll throw out the red ones“

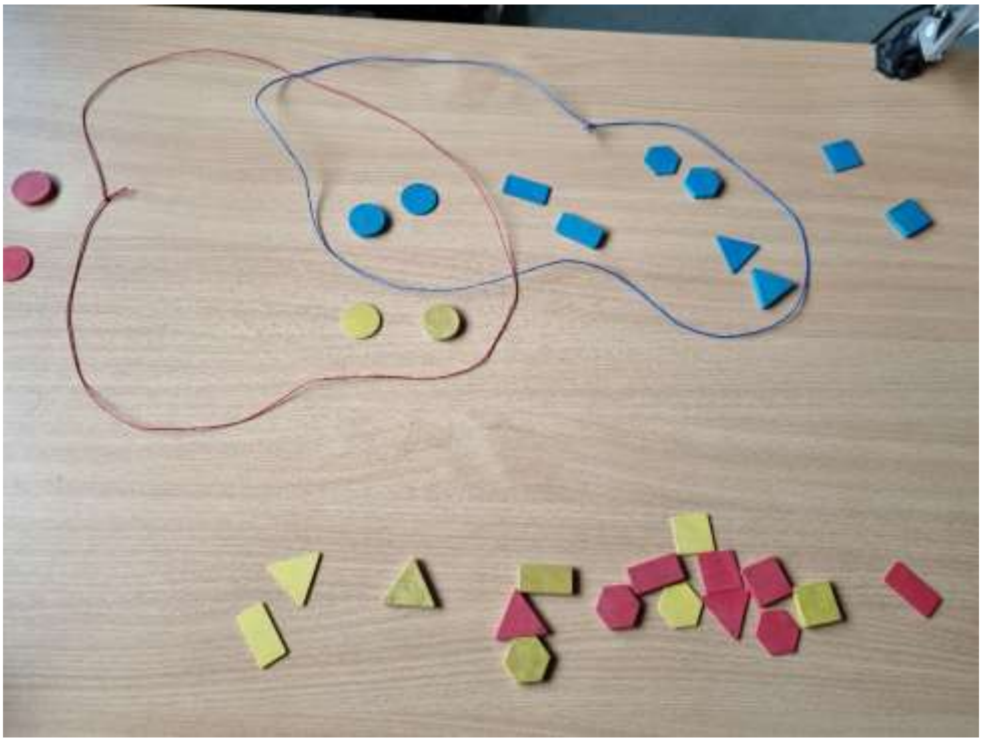


„Look Pilko, we got all the round figures that are not red“.

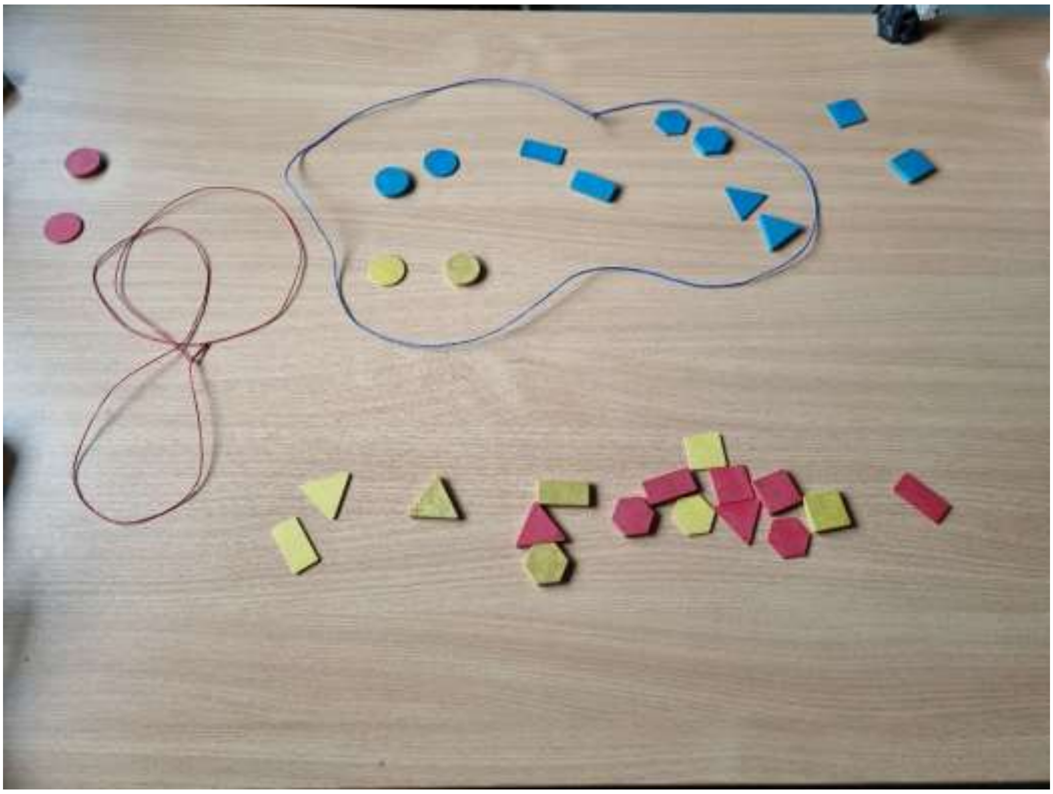
„In the same way we will find all blue figures that are not squares. First we will find all the blue figures"



„Now we will take among them those that are not squares, so we throw out the squares"



„You see, Pilko," Nina explained to him, „the figures you should have singled out are the figures that are round and not red, and those are the ones in the left fence, or they are blue and not square, and those are the ones in the right fence. So the round and not red or blue and not square figures are all these figures together. And that is the solution."



„Ha, now I understand“, said Pilko, although he wasn't quite sure yet whether he understood.

While Pilko looked at the solution in disbelief, the children and the bees reached a beautiful meadow full of yellow dandelions. The bees filled their bags and on their way back, the whole forest echoed with their happy song. When it was time to say goodbye, they asked the children how to thank them. The children answered that their greatest gratitude was to see them happily buzzing with nature. Then they hurried home, so that their parents wouldn't worry, because it was already getting dark.

That year, the orchards were full of fruit. Parents could not be surprised at how many there were. They said that there had never been so much fruit in any year. The children just smiled, because they knew the answer.

2 THE WOLF AND THE TEN LITTLE GOATS

©Boris

(sketch for a modified classic story in which the wolf was confused by the numbers)

The mother goat said to the little goats: „Dear children, we have run out of food. I have to go to the supermarket to get some. And our dishes were breaking. I will buy you 10 saucers and 10 glasses, so each of you will have your own saucer and your own glass.“ A wolf that was passing by heard it. After thinking for a long time, because he was not very good at counting, the wolf happily concluded that there were 20 little goats inside (a child's reaction is expected). Soon they will be alone and he will hunt them all down. What a feast it will be! As soon as the mother goat left, the wolf knocked on the door and said in a wolf's hoarse voice: „Open, children, your mother has come.“ Little goats stopped laughing: „What kind of mommy are you? You are a wolf. We know you by your voice.“ The angry wolf went to the stream and drank a lot of cold water to make his voice hoarse. He knocked on the door again and said in a sweet voice: „Open the door, dear children. Your mom came. I brought you all kinds of food to eat.". „Tell mother the password first“, answered the cautious little goats. „What password? What is the password?“, the wolf asked himself (we can ask the children about the password). „I forgot to buy the password“, said the wolf, and the little goats burst into laughter: „You're not our mom. You are a wolf.“ The wolf angrily bangs on the door. What a surprise! The door opened wide. Mama goat, making sure not to forget what she was going to buy, forgot to close the door! The little goats started screaming, and the wolf started collecting them in a sack. When he put all the kids in the sack, he decided to count them to see if any were missing. But it wasn't easy for him to do, because he didn't want to spread the sack so that some little goats wouldn't escape. That's why he put his hand in the bag and tried to count them by touching them. But the little goats squirmed, so once he counted 12 goats, the second time 8 and the third time 15. „It doesn't matter how many there are exactly. The important thing is that every time I got that there were more than 20 little goats“, the wolf thought with satisfaction (a child's reaction is expected). He barely managed to put the sack on his shoulders, it was too heavy. But he continued anyway, because he didn't want to throw out any kid and lighten the sack. On the way to the forest, he had to cross a stream over a log. At the beginning of the log there was a sign with the warning: „A maximum of 7 little goats can be transferred over the log!“. Since the wolf was not good at counting, he asked the kids if 20 is greater than 7. „It is“, shouted the wise kids. „You have to let one of us go, so there will be 7 of us“, the kids lied. The wolf lets one go, thinking that there are now 7 in the bag (here we can ask the children how many are in the bag now and if it is more than 7). The released kid quickly hurries to inform his mother. As there were now 9 of them in the bag, when the wolf crossed the log, the log broke (we can ask the children why this happened), the wolf fell into the stream, broke his leg, the bag fell on him, got loose and the goats scattered. Not to mention what happened to him when the mother goat came to get her kids and found him in the stream. He barely made it out alive.